UNIT- I

MULTIVARIABLE CALCULUS (INTEGRATION)

DOUBLE INTEGRATION

We know that the double integral over the region R of a function f(x,y) is $\iint_R f(x,y)dxdy$

Case(i)

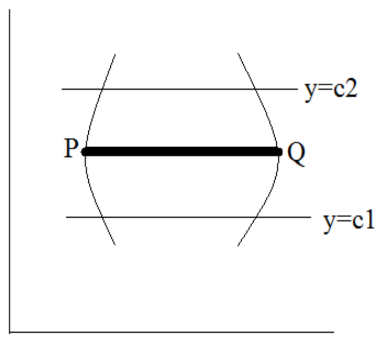
let R be the region bounded by the lines $x=c_1,\ x=c_2,\ y=c_3,\ y=c_4$ where c_1,c_2,c_3,c_4 are constant. Clearly the region *R is a rectangle*.

We know the region integration R, the double integral $\iint_R f(x,y)dxdy$ can be written as $\int_{c_3}^{c_4}\int_{c_1}^{c_2} f(x,y)dxdy$

Case (ii)

Consider the double integral $\int_{c_1}^{c_2}\int_{x_1}^{x_2} f(x,y)dxdy$

Suppose x_1 and x_2 are the function of y say $x_1 = f(y),\ x_2 = \phi(y)$ and c_1 *and* c_2 are constants then the region of integration *R is bounded by curve* $x_1 = f(y)$ and $x_2 = \phi(y)$ and the lines $y = c_1$ *and* $y = c_{2.}$



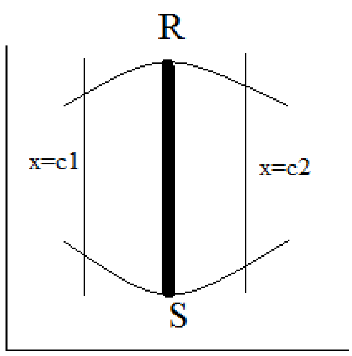
The region shown in figure. Here we integrate $f(x,y)$ first w.r.to x. Keeping y as a constant and the resulting expression w.r.to *y*.

i.e first integration is along the horizontal strip PQ and then slide this strip PQ vertically

Case(iii)

Consider the double integral $\int_{c_1}^{c_2}\int_{y_1}^{y_2} f(x,y)dydx$

Suppose y_1 and y_2 are the function of *x* say $y_1 = f(x),\ y_2 = \phi(x)$ and c_1 *and* c_2 are constants then the region of integration *R is bounded by curve* $y_1 = f(x)$ and $y_2 = \phi(x)$ and the lines $x = c_1$ *and* $x = c_{2.}$



The region shown in figure. Here we integrate $f(x,y)$ first w.r.to *y*. Keeping *x* as a constant and integrate the resulting expression w.r.to *x*.

i.e first integration is along the vertical strip RS and then slide this strip RS horizontally

Problem:-01.

Evaluate $\int_0^1 \int_1^2 x(x+y)dy\,dx$

Solution :-

Let I = $\int_0^1 \int_1^2 x(x+y)dy\,dx$

$= \int_{x=0}^{x=1} [\int_{y=1}^{y=2} x(x+y)dy]\,dx$

$= \int_0^1 x\left[xy + \frac{y^2}{2}\right]_{y=1}^{y=2} dx$

$= \int_0^1 x\left\{\left[x.2 + \frac{2^2}{2}\right] - \left[x.1 + \frac{1}{2}\right]\right\} dx$

$= \int_0^1 x\left(x + \frac{3}{2}\right) dx$

$= \int_0^1 \left(x^2 + \frac{3x}{2}\right) dx$

$= \left[\frac{x^3}{3} + \frac{3x^2}{2}\right]_{x=0}^{x=1}$

$= \frac{1}{4} + \frac{3}{4} = \frac{13}{12}$

Note:-

If all the limits of double integrals are numbers, then the integrals are indentified using rectangle box, and any order of integration (x first y second or yfirst x second) can be followed, both will give same answer.

Problem:-02

Evaluate $\int_2^3 \int_1^2 \frac{1}{xy}\,dx\,dy$

Solution:-

I = $\int_{y=2}^{y=3} [\int_{x=1}^{x=2} \frac{1}{xy}\,dx]\,dy$

$= \int_2^3 \frac{1}{y}\,(\log x)_{x=1}^{x=2} dy$

$= \int_2^3 \frac{1}{y}\,(\log 2 - \log 1)\,dy$

$= \int_2^3 \frac{1}{y} (\log 2)\, dy = \log 2 \, [\log y]_{y=2}^{y=3}$

= log 2 [log 3 – log 2]

= log2 . log 3/2

= log 2*3/2

= log 3.

Problem:- 3

Evaluate: $\int_{x=0}^{x=5} \int_{y=0}^{y=x^2} x(x^2 + y^2)\, dx\, dy$

Solution

Let I = $\int_0^5 \int_0^{x^2} x(x^2 + y^2)\, dx\, dy$

$= \int_0^5 \int_0^{x^2} (x^3 + xy^2)\, dy\, dx = \int_0^5 \left[x^5 + \frac{x^7}{3} \right] dx$

$= \int_0^5 \left[y\, x^3 + \frac{xy^3}{3} \right]_{y=0}^{y=x^2} dx$

$= \left[\frac{x^6}{6} + \frac{x^8}{8*3} \right]_0^5$

$= \left[\frac{5^6}{6} + \frac{5^8}{8*3} \right]$

$= 5^6 \left[\frac{1}{6} + \frac{25}{24} \right]$

$= 5^6 \left[\frac{29}{24} \right]$.

Note

If all the limits of double integrals are not constant, integral which has variable limit should be evaluated first.

Suppose if the integral limit is a function of x say f(x), then it is corresponding to y integral .i.e *y=f(x).* Therefore the order of integration *is y first x second.*

Suppose if the integral limit is a function of y say f(y), then it is corresponding to x integral .i.e *x=f(y).* Therefore the order of integration *is x first y second.*

Problem:-04

. Evaluate $\int_0^1 \int_0^y x^2 dy dx$

Solution:

$$\int_0^1 \int_0^y x^2 dy dx = \int_0^1 \left(\frac{x^3}{3}\right)_0^y dy$$

$$= \frac{1}{3}\int_0^1 y^3 dy$$

$$= \frac{1}{3}\left(\frac{y^4}{4}\right)_0^1$$

$$= \frac{1}{3}\left(\frac{1}{4}\right)$$

$$= \frac{1}{12}$$

Problem:-05

Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1+x^2+y^2}$

Solution:

Let I = $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\, dy}{1+x^2+y^2}$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{\left(\sqrt{1+x^2}\right)^2 + y^2}\, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} tan^{-1}(1) - tan^{-1}(0)\right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \frac{\pi}{4}\right] dx$$

$$= \frac{\pi}{4} \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \right] dx$$

$$= \frac{\pi}{4} \left[sinh^{-1}(x) \right]_0^1$$

$$= \frac{\pi}{4} \left[sinh^{-1}(1) - sinh^{-1}(0) \right]$$

$$= \frac{\pi}{4} \left[sinh^{-1}(1) - 0 \right]$$

$$= \frac{\pi}{4} \left[\log(1 + \sqrt{2} \right].$$

Problem:-06

Evaluate $\iint xy(x+y)dxdy$ over the area between $y = x^2, y = x$

Solution:

Given

$x^2 = y \quad and \quad y = x$

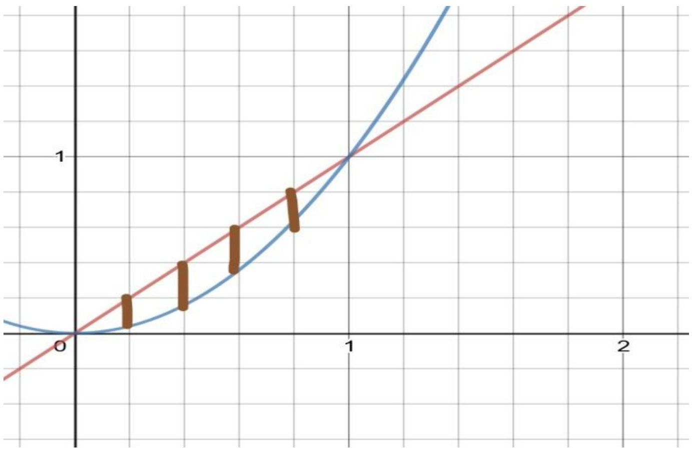
$x^2 = x \Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0, \; x = 1$

$put \; x = 0 \; we \; get \; y = 0$

$put \; x = 1 \; we \; get \; y = 1$



The intersecting points are (0,0) and (1,1)

Hence the limits are

x=0	*x=1*
$y=x^2$	*y=x*

$$\iint xy(x+y)dxdy = \int_0^1 \int_{x^2}^{x} xy(x+y)dydx$$

$$= \int_0^1 \int_{x^2}^{x} (x^2y + xy^2)dydx$$

$$= \int_0^1 \left(\frac{x^2y^2}{2} + \frac{xy^3}{3}\right)_{x^2}^{x} dx$$

$$= \int_0^1 \left(\left(\frac{x^4}{2} + \frac{x^4}{3}\right) - \left(\frac{x^6}{2} + \frac{x^7}{3}\right)\right)dx$$

$$= \int_0^1 \left(\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right)dx$$

$$= \left(\frac{5x^5}{30} - \frac{x^7}{14} - \frac{x^8}{24}\right)_0^1$$

$$= \left(\frac{5}{30} - \frac{1}{14} - \frac{1}{24}\right)$$

$$= \left(\frac{1}{6} - \frac{1}{14} - \frac{1}{24}\right)$$

$$= \frac{84 - 36 - 21}{504}$$

$$= \frac{27}{504}$$

$$= \frac{3}{56}$$

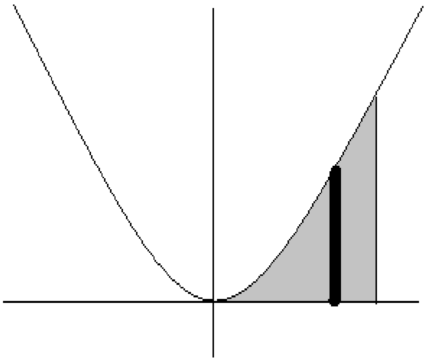
Problem:-07

Evaluate $\iint_A xydxdy$ where *A* is the region bounded by x = 2a and the curve $x^2 = 4ay$.

Solution:-

Given that *x = 2a*. In the figure *x* varies from *x = 0* to *x = 2a*. To find the limit for *y*, we take a strip PQ parallel to the y – axis, it's lower end P lies on *y = 0* and upper end Q lies on

$x^2 = 4ay$ ⇒ $y = \frac{x^2}{4a}$



$$\int\int xy\,dx\,dy = \int_{x=0}^{x=2a}\int_{y=0}^{y=x^2/4a} xy\,dy\,dx$$

$$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \frac{1}{2}\int_0^{2a} x\left[\frac{x^4}{16\,a^2}\right] dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\ dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= \frac{1}{32a^2}\left[\frac{2^6a^6}{6}\right]$$

$$= \frac{a^4}{3}$$

CHANGE OF ORDER OF INTEGRATION

The evaluation of some double integrals may be very difficult. In this case we may evaluate it easily by changing the order of integration in a given double integral. When we change the order of integration the limits are also changed but the there will be no change in final answer. The following points are very important when the change of order of integration takes place.

(i) If the limits of the inner integral is a function of *x* (or function of *y*) the first integration should be w.r.to *y* (or w.r.to *x*)

(ii) Draw the region of integration by using the given limits.

(iii) If the integration is first w.r.to *x* keeping *y* as a constant then consider the vertical strip and find the new limits accordingly

(iv) If the integration w.r.to *y* keeping *x* as a constant then consider the horizontal strip and find the new limits accordingly

(v) After finding the new limits evaluate the inner integral first and then the outer integral

PROBLEMS ON CHANGE OF ORDER OF INTEGRATION

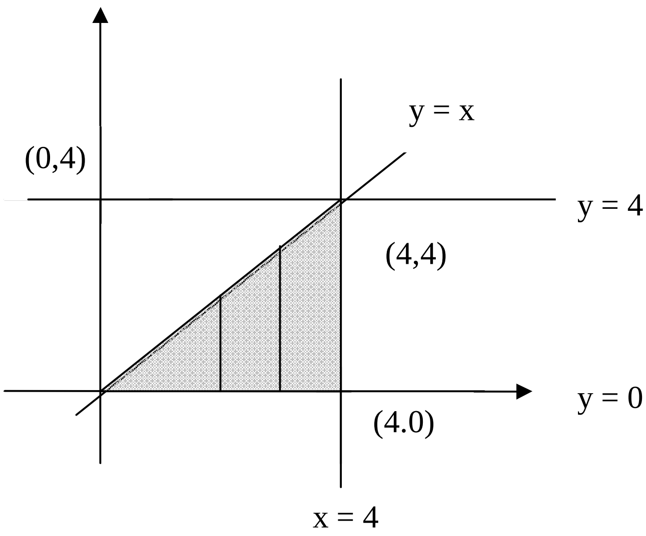
Problem:-01

Change the order of integration and evaluate $\int_0^4 \int_y^4 \frac{x}{x^2+y^2}\, dy\, dx$

Solution:

Rewriting the given integral in proper order, we have $\int_0^4 \int_y^4 \frac{x}{x^2+y^2}\, dx\, dy$

$\therefore$ The region of integration is bounded by $x = y \quad y = 0$; $x = 4 \quad y = 4$



y=0; y=x

x=0; x=4

Given integral limits are corresponds to horizontal strip method, So By changing the order, we have consider vertical strip method

$$I = \int_0^4 \int_0^x \frac{x}{x^2+y^2}\, dx\, dy$$

$$= \int_0^4 \left(\tan^{-1} \frac{y}{x} \right)_0^x dx$$

$$= \int_0^4 \left(\tan^{-1} 1 - \tan^{-1} 0 \right) dx$$

$$= \frac{\pi}{4} \int_0^4 dx$$

$$= \frac{\pi}{4} [x]_0^4$$

$$= \frac{\pi}{4} [4 - 0]$$

$$= \frac{\pi}{4} (4)$$

$$= \pi$$

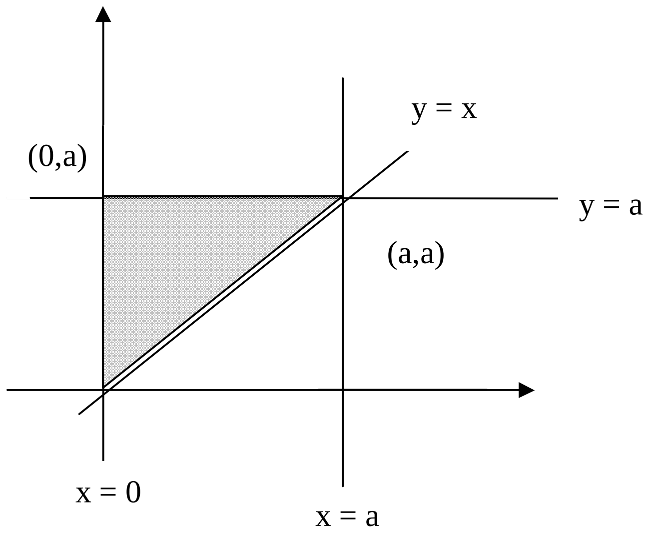
Problem :-02.

Change the order of integration and evaluate the integral $\int_0^a \int_x^a \left(x^2 + y^2 \right) dy\, dx$

Solution:

Given integral is in proper form.

$\therefore$ The region of integration is bounded by $x = 0$, $x = a$, $y = x$, $y = a$



By changing the order, we have

$$I = \int_0^a \int_0^y \left(x^2 + y^2\right) dx\, dy$$

$$= \int_0^a \left(\frac{x^3}{3} + y^2 x\right)_0^y dy$$

$$= \int_0^a \left(\frac{y^3}{3} + y^3\right) dy$$

$$= \frac{4}{3}\int_0^a y^3 dy$$

$$= \frac{4}{3}\left[\frac{y^4}{4}\right]_0^a = \frac{4}{3}\frac{a^4}{4} = \frac{a^4}{3}$$

Problem:-3

Change the order of integration $\int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} dy\, dx$ and evaluate it.

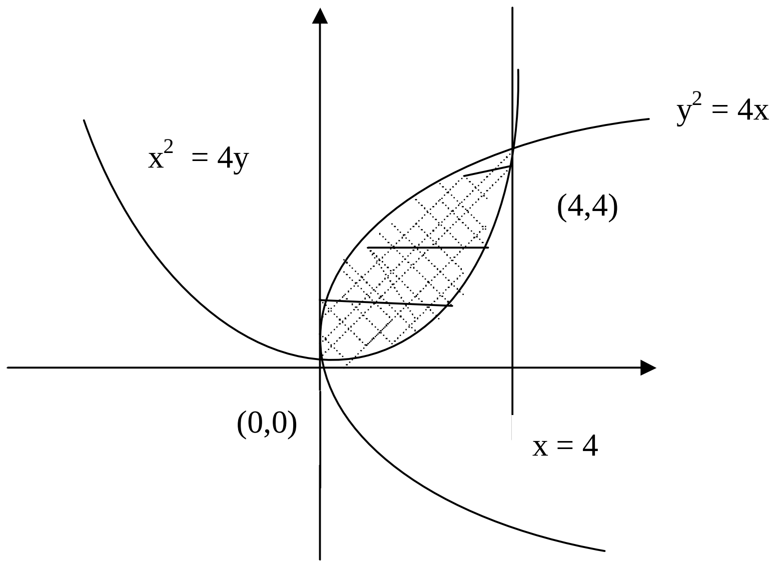
Solution:

Given integral is in proper form.

$\therefore$ The region of integration is bounded by

$x = 0 \qquad y = \frac{x^2}{4} \qquad i.e.\ x^2 = 4y$

$x = 4 \qquad y = 2\sqrt{x} \qquad i.e.\ y^2 = 4x$



$x=y^2/4;\ x=2y^{1/2}$

$y=0, y=4$:

By changing the order, we have

$$I = \int_0^4 \int_{\frac{y^2}{4}}^{2\sqrt{y}} dx\,dy$$

$$= \int_0^4 2\sqrt{y} - \frac{y^2}{4}\,dy$$

$$= \int_0^4 2y^{\frac{1}{2}} - \frac{y^2}{4}\,dy$$

$$= \left[\frac{4}{3}y^{\frac{3}{2}} - \frac{y^3}{12}\right]_0^4$$

$$= \left[\frac{4}{3}4^{\frac{3}{2}} - \frac{64}{12}\right]$$

$$= \frac{16}{3}$$

Problem:-4

Change the order of integration and hence evaluate $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy \; dy \, dx$

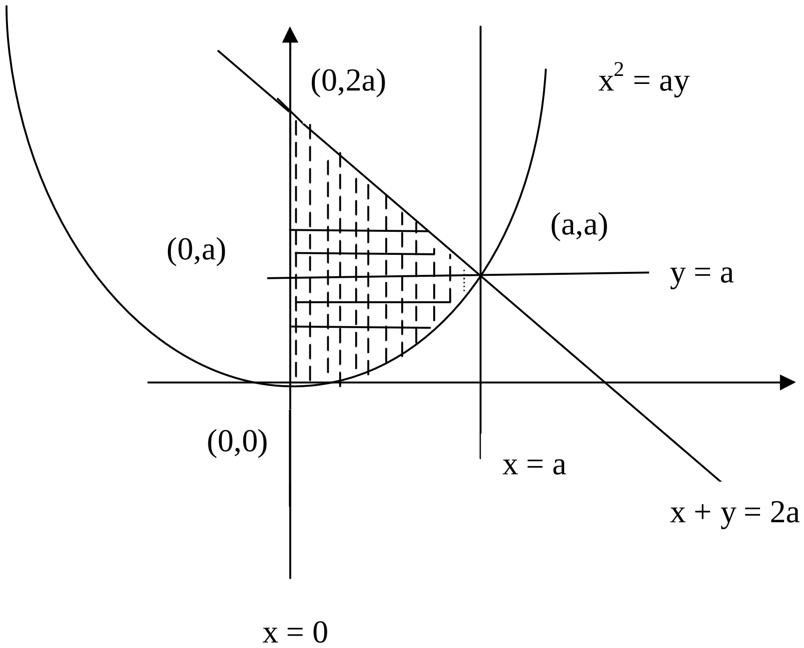
Solution:

Given integral is in proper form.

$\therefore$ The region of integration is bounded by

$x = 0 \qquad y = \frac{x^2}{a} \qquad i.e. \qquad x^2 = ay$

$x = a \qquad y = 2a - x \qquad i.e. \qquad x + y = 2a$



By changing the order, we have $R=R_1+R_2$

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy \; dy \, dx$$

R1

x=0;   x=(ay)$^{1/2}$

y=0;y=a

R2

x=0; x=y-2a

y=a to y=2a.

$$I = \int_0^a \int_0^{\sqrt{a\,y}} xy\ dx\,dy + \int_1^{2a} \int_0^{2a-y} xy\ dx\,dy$$

$$= \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{a\,y}} dy + \int_a^{2a} y\left(\frac{x^2}{2}\right)_0^{2a-y} dy$$

$$= \frac{a}{2}\int_0^a y^2\,dy + \frac{1}{2}\int_a^{2a} y\left(2a-y\right)^2 dy$$

$$= \frac{a}{2}\int_0^a y^2\,dy + \frac{1}{2}\int_a^{2a} 4a^2y+y^3-4ay^2\,dy$$

$$= \frac{a}{2}\left(\frac{y^3}{3}\right)_0^a + \frac{1}{2}\left(2ay^2+\frac{y^4}{4}-\frac{4ay^3}{3}\right)_a^{2a}$$

$$= \frac{a^4}{6}+\frac{1}{2}\left[\left(8a^4+4a^4-\frac{32a^4}{3}\right)-\left(2a^4+\frac{a^4}{4}-\frac{4a^4}{3}\right)\right]$$

$$= \frac{9}{24}a^4$$

Problem:-5.

Evaluate by changing the order of integration $\int_0^a \int_0^{x^2} x(x^2+y^2)dydx$

Solution:

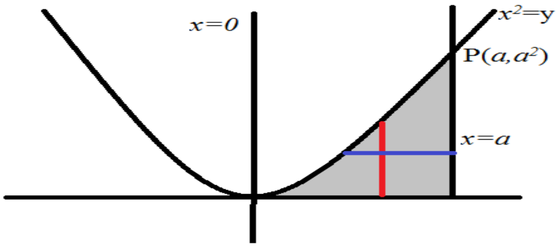
Given limits are

$y = 0 \quad (x-axis)$

$y = x^2 \quad (Parabola - vertex\ at\ origin, open\ upward)$

$x = 0 \quad (y-axis)$

$x = a \quad (st.line\ parallal\ y-axis)$



After the Change the order the limits are

$x=\sqrt{y}$

$x=a$

$y=0$

$y=a^2$

$$\int_0^a\int_0^{x^2} x(x^2+y^2)dydx=\int_0^{a^2}\int_{\sqrt{y}}^{a} x(x^2+y^2)dxdy$$

$$=\int_0^{a^2}\int_{\sqrt{y}}^{a}(x^3+xy^2)dxdy$$

$$=\int_0^{a^2}\left(\frac{x^4}{4}+\frac{x^2y^2}{2}\right)_{\sqrt{y}}^{a}dy$$

$$=\int_0^{a^2}\left\{\left(\frac{a^4}{4}+\frac{a^2y^2}{2}\right)-\left(\frac{y^2}{4}+\frac{y^3}{2}\right)\right\}dy$$

$$=\left(\frac{a^4y}{4}+\frac{a^2y^3}{6}-\frac{y^3}{12}-\frac{y^4}{8}\right)_0^{a^2}$$

$$=\frac{a^6}{4}+\frac{a^8}{6}-\frac{a^6}{12}-\frac{8^4}{8}$$

$$=\frac{6a^6+4a^8-2a^6-3a^8}{24}$$

$$=\frac{4a^6+a^8}{24}$$

$$=\frac{a^6(a^2+4)}{24}$$

Problem:-06

Evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}} xydydx$ by changing the order of integration

Solution:

Given limits are

$$y = 0 \quad (x - axis)$$
$$y = \sqrt{a^2 - x^2} \quad \Rightarrow y^2 = a^2 - x^2 \Rightarrow x^2 + y^2 = a^2 \quad (circle)$$
$$x = 0 \quad (y - axis)$$
$$x = a \quad (st.line\ parallal\ y - axis)$$

After the Change the order the limits are

$$x = 0 \qquad x = \sqrt{a^2 - y^2}$$
$$y = 0 \qquad y = a$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xydydx = \int_0^a \int_0^{\sqrt{a^2-y^2}} xydxdy$$
$$= \int_0^a y\left(\frac{x^2}{2}\right)_0^{\sqrt{a^2-y^2}} dy$$
$$= \frac{1}{2}\int_0^a y\left(a^2 - y^2\right)dy$$
$$= \frac{1}{2}\int_0^a \left(a^2 y - y^3\right)dy$$
$$= \frac{1}{2}\left(\frac{a^2 y^2}{2} - \frac{y^4}{4}\right)_0^a$$
$$= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$
$$= \frac{a^4}{8}$$

Problem:-7

Evaluate by changing the order of integration $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\ dy\, dx$

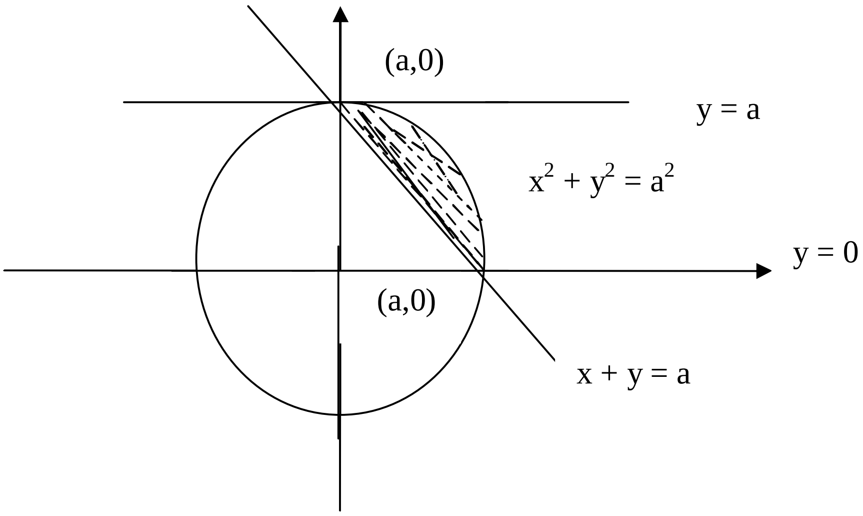
Solution:

Rewriting the given integral in proper order, we have $\int_0^a \int_{a-y}^{\sqrt{a^2-y^2}} y\ dx\, dy$

$\therefore$ The region of integration is bounded by

$y = 0 \qquad x = a - y \qquad i.e. \quad x + y = a$

$y = a \qquad x = \sqrt{a^2 - y^2} \qquad i.e. \quad x^2 + y^2 = a^2$



By changing the order, we have

$$I = \int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} y \; dy \, dx$$

$$= \frac{1}{2}\int_0^a \left[y^2 \right]_{a-x}^{\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_0^a \left(a^2 - x^2\right) - (a-x)^2 \, dx$$

$$= \frac{1}{2}\left[a^2 x - \frac{x^3}{3} - \frac{(a-x)^3}{-3} \right]_0^a$$

$$= \frac{1}{2}\left[a^3 - \frac{a^3}{3} - \frac{a^3}{3} \right]$$

$$= \frac{a^3}{6}$$

Problem:-8

Change the order of integration and hence evaluate $\int_0^1 \int_{y^2}^{2-y} xy\ dx\,dy$

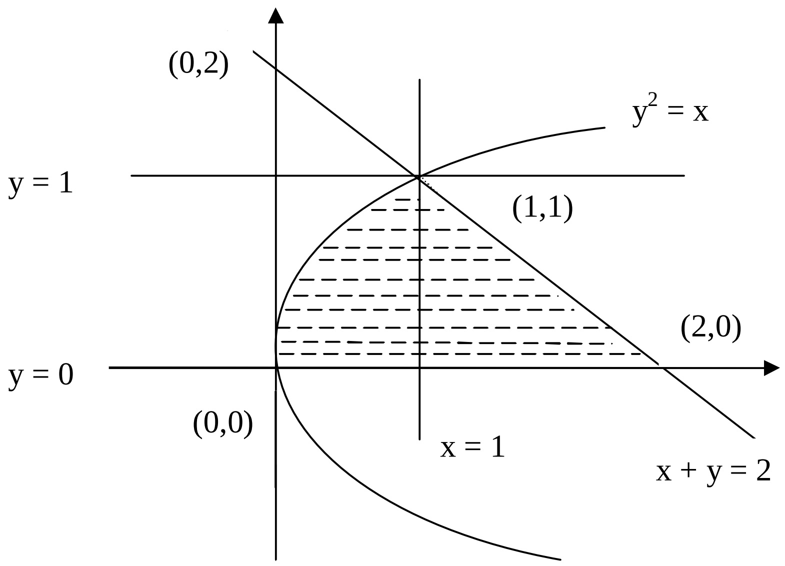
Solution:

Given integral is in proper form.

$\therefore$ The region of integration is bounded by

$y=0 \qquad x=y^2$

$y=1 \qquad x=2-y \qquad i.e.\ x+y=2$



By changing the order, we have

$$I = \int_0^1 \int_0^{\sqrt{x}} xy\ dy\,dx + \int_1^2 \int_0^{2-x} xy\ dy\,dx$$

$$= \int_0^1 x\left(\frac{y^2}{2}\right)_0^{\sqrt{x}} dx + \int_1^2 x\left(\frac{y^2}{2}\right)_0^{2-x} dx$$

$$= \frac{1}{2}\int_0^1 x^2\ dx + \frac{1}{2}\int_1^2 x(2-x)^2\ dx$$

$$= \frac{1}{2}\int_0^1 x^2\ dx + \frac{1}{2}\int_1^2 4x+x^3-4x^2\ dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3}\right)_0^1 + \frac{1}{2}\left(2x^2+\frac{x^4}{4}-\frac{4x^3}{3}\right)_1^2$$

$$= \frac{1}{6}+\frac{1}{2}\left[\left(8+4-\frac{32}{3}\right)-\left(2+\frac{1}{4}-\frac{4}{3}\right)\right]$$

$$= -\frac{23}{12}$$

TRIPLE INTEGRAL

Consider a function *f(x,y,z)* defined at every point of the three dimensional finite region V. divide V into *n* elementary volumes $\delta V_1, \delta V_2, ...\delta V_n$. Let (x_r, y_r, z_r) be any point within the *r*th sub-division δV_r.

Consider the sum $\sum_{r=1}^{\infty} f(x_r, y_r, z_r)\delta V_r$

The limits of this sum, if it exists, as $n \to \infty$ and $\delta V_r \to \infty$ is called the triple integral of *f(x,y,z)* over the region V and is denoted by

$$\iiint f(x, y, z)dV$$

For purposes of evaluation it can also be expressed as the repeated integral

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{x_z}^{z_2} f(x, y, z)dxdydz$$

If x_1, x_2 are constants: y_1, y_2 are either constants or functions of x and z_1, z_2 are either constants or funtions of x and y then this integral is evaluated as folloes:

First *f(x,y,z)* is integrated w.r.to z between the limits z_1 *and* z_2 keeping x and y fixed. The resulting expression is integrated w.r.to y between the limits y_1 *and* y_2 keeping x constant. The result just obtained is finally integrated w.r.to x between the limits x_1 *and* x_2

Thus $$I = \boxed{\int_{x_1}^{x_2}\boxed{\int_{y_1(x)}^{y_2(x)}\boxed{\int_{z_1(x,y)}^{z_2(x,y)} f(x, y, z)\ dz}\ dy}\ dx}$$

Where the integration is carried out from the innermost rectangle to the outermost rectangle.

Problem:-01

Evaluate $\int_0^a \int_0^b \int_0^c (x+y+z)\, dzdydx$

Solution:

$$\int_0^a \int_0^b \int_0^c (x+y+z)\, dzdydx = \int_0^a \int_0^b \left(xz + yz + \frac{z^2}{2} \right)_0^c dydx$$

$$= \int_0^a \int_0^b \left(cx + cy + \frac{c^2}{2} \right) dydx$$

$$= \int_0^a \left(cxy + c\frac{y^2}{2} + \frac{c^2}{2} y \right)_0^b dx$$

$$= \int_0^a \left(bcx + \frac{b^2c}{2} + \frac{bc^2}{2} \right) dx$$

$$= \left(bc\frac{x^2}{2} + \frac{b^2c}{2} x + \frac{bc^2}{2} x \right)_0^a$$

$$= \left(bc\frac{a^2}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2} \right)$$

$$= \left(\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2} \right)$$

$$= \frac{abc}{2}(a+b+c)$$

Problem :-2

Evaluate $\int_0^{\log a} \int_0^{x} \int_0^{x+\log y} \left(e^{x+y+z}\right) dzdydx$

Solution:

$$\int_0^{\log a} \int_0^{x} \int_0^{x+\log y} \left(e^{x+y+z}\right) dzdydx = \int_0^{\log a} \int_0^{x} \left(e^{x+y+z}\right)_0^{x+\log y} dydx$$

$$= \int_0^{\log a}\int_0^{x} \left(e^{2x+y+\log y} - e^{x+y}\right)dydx$$

$$= \int_0^{\log a}\int_0^{x} \left(e^{2x} ye^{y} - e^{x}e^{y}\right)dydx$$

$$= \int_0^{\log a} \left[e^{2x}\left(ye^{y} - e^{y}\right) - e^{x}e^{y}\right]_0^x dx$$

$$= \int_0^{\log a} \left[(e^{2x}\left(xe^{x} - e^{x}\right) - e^{x}e^{x}) - (e^{2x}\left(0 - e^{0}\right) - e^{x}e^{0})\right]dx$$

$$= \int_0^{\log a} \left[xe^{3x} - e^{3x} + e^{2x} - e^{2x} + e^{x}\right]dx$$

$$= \left[x\frac{e^{3x}}{3} - \frac{4e^{3x}}{9} + e^{x}\right]_0^{\log a}$$

$$= \left[\log a\frac{e^{3\log a}}{3} - 0 - \frac{4(e^{3\log a)} - e^{0})}{9} + e^{\log a} - e^{0}\right]$$

$$= \left[\frac{a^3}{3}\log a - \frac{4a^3}{9} + a - \frac{5}{9}\right]$$

Problem:-03.

Evaluate $\int_0^a \int_0^b \int_0^c xyz\, dz\, dy\, dx$

Solution:-

$$\int_0^a \int_0^b \int_0^c xyz\, dz\, dy\, dx = \int_0^a x\, dx \int_0^b y\, dy \int_0^c z\, dz$$

$$= \left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b \left[\frac{z^2}{2}\right]_0^c$$

$$= \left[\frac{a^2}{2} - 0\right]\left[\frac{b^2}{2} - 0\right]\left[\frac{c^2}{2} - 0\right]$$

$$= \frac{(abc)^2}{8}$$

Problem :-04

Evaluate $\int_0^a \int_0^b \int_0^b (x^2 + y^2 + z^2)\, dz\, dy\, dx$

Solution:-

Given I $= \int_0^a \int_0^b \int_0^b (x^2 + y^2 + z^2)\, dz\, dy\, dx$

$= \int_0^a \int_0^b \left[\frac{x^3}{3} + xy^2 + xz^2\right]_0^c dy\, dz$

$= \int_0^a \int_0^b \left[\frac{c^3}{3} + cy^2 + cz^2\right] dy\, dz$

$= \int_0^a \left[\frac{c^3}{3} y + \frac{y^3}{3} c + cz^2 y\right]_0^b dz$

$= \int_0^a \left[\frac{c^3}{3} b + \frac{b^3}{3} c + cz^2 b\right] dz$

$= \left[\frac{c^3}{3} bz + \frac{b^3}{3} cz + c\frac{z^3}{3} b\right]_0^a$

$= \left[\frac{c^3}{3} ba + \frac{b^3}{3} ca + c\frac{a^3}{3} b\right]$

$= \frac{abc}{3}(a^2 + b^2 + c^2)$

Problem:-05

Evaluate: $\int_0^2 \int_0^3 \int_0^2 xy^2\, z\, dz\, dy\, dx$

Solution

Given that I= $\int_0^2 \int_0^3 \int_0^2 xy^2\, z\, dz\, dy\, dx$

$= \int_0^2 x\, dx \int_0^3 y^2\, dy \int_0^2 z\, dz$

$= \left[\frac{x^2}{2}\right]_0^2 \left[\frac{y^3}{3}\right]_1^3 \left[\frac{z^2}{2}\right]_1^2$

$= \left[\frac{4}{2} - 0\right]\left[\frac{27}{3} - \frac{1}{3}\right]\left[\frac{4}{2} - \frac{1}{2}\right]$

$= 26$

Problem:-06

Evaluate $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dzdydx$

Solution:-

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dzdydx = \int_0^1 \int_0^{1-x} [z]_0^{1-x-y} dydx$$

$$= \int_0^1 \int_0^{1-x} [1-x-y] dydx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right] dx$$

$$= \int_0^1 \left[(1-x)(1-x) - \frac{(1-x)^2}{2} \right] dx$$

$$= \int_0^1 \left[\frac{(1-x)^2}{2} \right] dx$$

$$= \left[\frac{(1-x)^3}{-6} \right]_0^1$$

$$= \frac{1}{6}$$

Problem:-07

Evaluate $\iiint \frac{dzdydx}{(1+x+y+z)^3}$ over the region bounded by *x=0*, *y=0* ,z=0 and *x+y+z=1*

Solution:-

The region is *x=0*, *y=0* and *x+y+z=1*

Hence the limits are

x=0	*x=1 (put y=0, z=0)*
y=0	*y=1-x (put z=0)*
z=0	*z=1-y-z*

$$\iiint \frac{dzdydx}{(1+x+y+z)^3} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dzdydx}{(1+x+y+z)^3}$$

$$=\int_0^1\int_0^{1-x}\int_0^{1-x-y}(1+x+y+z)^{-3}\,dzdydx$$

$$=\int_0^1\int_0^{1-x}\left(\frac{(1+x+y+z)^{-2}}{-2}\right)_0^{1-x-y}dydx$$

$$=-\frac{1}{2}\int_0^1\int_0^{1-x}\left[(2)^{-2}-(1+x+y)^{-2}\right]dydx$$

$$=-\frac{1}{2}\int_0^1\int_0^{1-x}\left[\frac{1}{4}-(1+x+y)^{-2}\right]dydx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}y+(1+x+y)^{-1}\right]_0^{1-x}dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}(1-x)+(2)^{-1}-(1+x)^{-1}\right]dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}(1-x)+\left(\frac{1}{2}\right)-\frac{1}{1+x}\right]dx$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}(x-\frac{x^2}{2})+\left(\frac{1}{2}x\right)-\log(1+x)\right]_0^1$$

$$=-\frac{1}{2}\int_0^1\left[\frac{1}{4}(1-\frac{1}{2})+\left(\frac{1}{2}\right)-\log(2)\right]$$

$$=\frac{1}{2}\log 2-\frac{5}{16}$$

Problem:-08

Evaluate $\iiint\frac{dzdydx}{\sqrt{a\ -x^2-y^2-z^2}}$ over the first octant of the sphere.

Solution:-

The equation of sphere $x^2+y^2+z^2=a^2$ and the limits are

$x=0$ $\qquad$ $x=a$

$y=0$ $\qquad$ $y=\sqrt{a^2-x^2}$

$z=0$ $\qquad$ $z=\sqrt{a^2-x^2-y^2}$

$$\iiint\frac{dzdydx}{\sqrt{a^2-x^2-y^2-z^2}}=\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}}\frac{dzdydx}{\sqrt{(a^2-x^2-y^2)-z^2}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{(a^2-x^2-y^2)}} \right) \right]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0) \right] dydx$$

$$= \frac{\pi}{2} \int_0^a \int_0^{\sqrt{a^2-x^2}} dydx$$

$$= \frac{\pi}{2} \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^a \sqrt{a^2-x^2}\, dx$$

$$= \frac{\pi}{2} \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a} \right) \right]_0^a$$

$$= \frac{\pi}{2} \left[0 + \frac{a^2}{2} \sin^{-1}(1) - 0 \right]$$

$$= \frac{\pi}{2} \left[\frac{a^2}{2} \frac{\pi}{2} \right]$$

$$= \frac{\pi^2 a^2}{8}$$

GREEN'S THEOREM

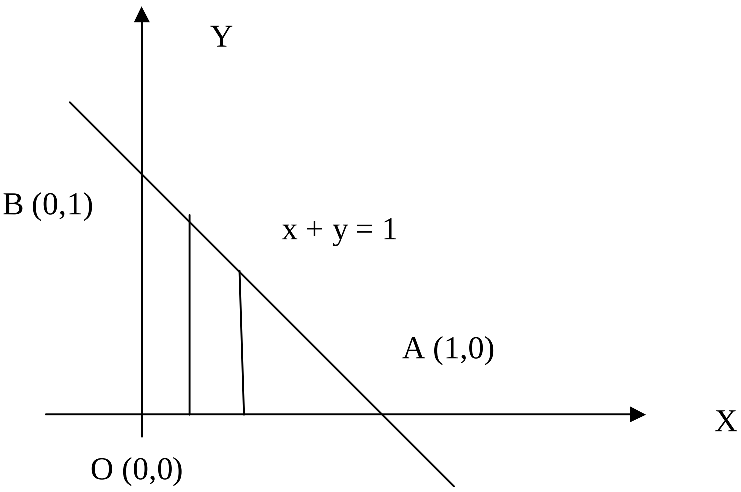
If u, v, $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ are continuous and single valued functions in the region R enclosed by the curve C, then $\int_C u\, dx + v\, dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy$

Problem:-01

Verify Green's theorem, in plane for $\int_C (3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy$ where C is the boundary of the triangle formed by the lines x = 0, y = 0 and x + y = 1 in the xy plane.

Solution:-

Green's theorem is $\int_C u\, dx + v\, dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy$



$u = 3x^2 - 8y^2 \qquad v = 4y - 6xy$

$\frac{\partial u}{\partial y} = -16y \qquad \frac{\partial v}{\partial x} = -6y$

The limits are

y=0 *y=1-x*

x=0 *x=1*

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy = \int_0^1 \int_0^{1-x} (-6y + 16y)\, dydx$$

$$= \int_0^1 \int_0^{1-x} 10y dydx$$

$$= 10 \int_0^1 \left(\frac{y^2}{2} \right)_0^{1-x}$$

$$= 5 \int_0^1 (1-x)^2$$

$$= 5 \left(\frac{(1-x)^3}{-3} \right)_{0\leq}^{1}$$

$$= \frac{5}{3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\int_C u\ dx\ +\ v\ dy = \int_{OA} u\ dx\ +\ v\ dy + \int_{AB} u\ dx\ +\ v\ dy + \int_{BO} u\ dx\ +\ v\ dy$$

Along OA, $y = 0$ and hence $dy = 0$. Also x varies from 0 to 1.

$$\therefore \int_{OA} \left(3x^2 - 8y^2\right)dx + (4y - 6xy)\,dy = \int_0^1 3x^2\,dx = \left[x^3\right]_0^1 = 1$$

Along AB, $x + y = 1 \; or \; x = 1 - y$ and hence $dx = -dy$. Also y varies from 0 to 1.

$$\therefore \int_{BO} \left(3x^2 - 8y^2\right)dx + (4y - 6xy)\,dy = \int_{AB} -\left(3(1-y)^2 - 8y^2\right)dy + (4y - 6y(1-y))\,dy$$

$$= \int_0^1 11y^2 + 4y - 3dy = \left[11\frac{y^3}{3} + 2y^2 - 3y\right]_0^1 = \frac{11}{3} + 2 - 3 = \frac{8}{3}$$

Along BO, $x = 0$ and hence $dx = 0$. Also y varies from 1 to 0.

$$\therefore \int_{BO} \left(3x^2 - 8y^2\right)dx + (4y - 6xy)\,dy = \int_1^0 4y\,dy = \left[2y^2\right]_1^0 = -2$$

$$\therefore \int_C \left(3x^2 - 8y^2\right)dx + (4y - 6xy)\,dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \quad \text{........................(2)}$$

From (1) and (2)

Green's theorem is verified

Problem:-02

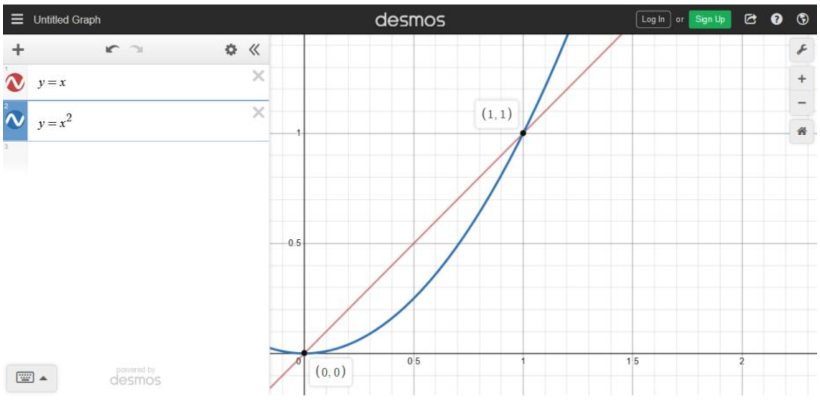
Verify Green's theorem in the plane for $\oint \left(xy + y^2\right)dx + x^2 dy$ where C is the region bounded by *y=x* and *y=x*²

Solution:-

Green's theorem is $\int_C u\;dx \;+\; v\;dy \;=\; \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\;dy$

$$u = xy + y^2 \qquad\qquad v = x^2$$

$$\frac{\partial u}{\partial y} = x + 2y \qquad\qquad \frac{\partial v}{\partial x} = 2x$$



The point of intersection of $y=x^2$ and $y=x$ are

(0,0) and (1,1) and the limits are

$x=0$	$x=1$
$y=x^2$	$y=x$

$$\iint_R \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)dxdy=\int_0^1\int_{x^2}^{x}(2x-x-2y)\,dydx$$

$$=\int_0^1\int_{x^2}^{x}(x-2y)\,dydx$$

$$=\int_0^1\left(xy-y^2\right)_{x^2}^{x}dx$$

$$=\int_0^1\left[\left(x^2-x^2\right)-\left(x^3-x^4\right)\right]dx$$

$$=\int_0^1\left(x^4-x^3\right)dx$$

$$=\left(\frac{x^5}{5}-\frac{x^4}{4}\right)_0^1$$

$$=\frac{1}{5}-\frac{1}{4}$$

$$=-\frac{1}{20} \qquad \text{.....................................(1)}$$

$$\int_C u\ dx\ +\ v\ dy=\int_{OA} u\ dx\ +\ v\ dy+\int_{AO} u\ dx\ +\ v\ dy$$

Along OA $y=x^2$, $dy=2xdx$

$$\int_{OA} u\ dx\ +\ v\ dy = \int_{OA} (xy+y)dx + x^2 dy$$

$$= \int_0^1 \left[\left(x^3+x^4\right)+2x^3\right]dx$$

$$= \left[\frac{x^4}{4}+\frac{x^5}{5}+\frac{2x^4}{4}\right]_0^1$$

$$= \left(\frac{x^5}{5}+\frac{3x^4}{4}\right)_0^1$$

$$= \frac{1}{5}+\frac{3}{4}$$

$$= \frac{19}{20}$$

Along AO, $y = x$, $dy = dx$

$$\int_{AO} u\ dx\ +\ v\ dy = \int_{AO} (xy+y)dx + x^2 dy$$

$$= \int_1^0 \left[x^2+x^2+x^2\right]dx$$

$$= \int_1^0 \left[3x^2\right]dx$$

$$= 3\left(\frac{x^3}{3}\right)_1^0$$

$$= -1$$

$$\int_C u\ dx\ +\ v\ dy = \frac{19}{20}-1$$

$$= -\frac{1}{20} \quad ....(2)$$

GAUSS DIVERGENCE THEOREM.

The surface integral of the normal component of a vector function f over a closed surface S enclosing volume V is equal to the volume integral of the divergence of f taken throughout the volume V.

i.e. $\iint_S \vec{f}.\hat{n}\ ds\ =\ \iiint_V \nabla.\vec{f}\ dV$

Problem:-01

Verify Gauss Divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ over the volume of the cuboid formed by the planes x = 0, x = a, y = 0, y = b, z = 0, z = c.

Solution:-

Gauss Divergence Theorem is $\iint_S \vec{f}.\hat{n}\, ds = \iiint_V \nabla.\vec{f}\, dV$

$$\nabla.\vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right) = 2x + 2y + 2z$$

$$\iiint_V \nabla.\vec{f}\, dV = \int_0^a \int_0^b \int_0^c (2x+2y+2z)\, dz\, dy\, dx$$

$$= \int_0^a \int_0^b [2xz + 2yz + z^2]_0^c\, dy\, dx$$

$$= \int_0^a \int_0^b [2xc + 2yc + c^2]\, dy\, dx$$

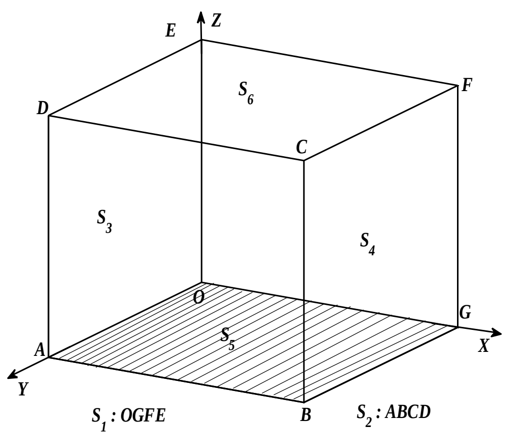
$$= \int_0^a [2xcy + y^2 c + yc^2]_0^b\, dx$$

$$= \int_0^a [2xcb + b^2 c + bc^2]\, dx$$

$$= [x^2 cb + xb^2 c + xbc^2]_0^a$$

$$= [a^2 cb + ab^2 c + abc^2]$$

$$= abc[a + b + c] \quad .....(1)$$



To evaluate $\iint_S \vec{f}.\hat{n}\, ds$ where $S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$

On $S_1 : OGFE\ \ y = 0\ \&\ \hat{n} = -\vec{j}$ and hence $\vec{f}.\hat{n} = -y^2 = 0$

$$\therefore \iint_{S_1} \vec{f}.\hat{n}\, ds = 0$$

On $S_2 : ABCD\ \ y = b\ \&\ \hat{n} = \vec{j}$ and hence $\vec{f}.\hat{n} = y^2 = b^2$

$$\therefore \iint_{S_2} \vec{f}.\hat{n}\, ds = \int_0^c \int_0^a b^2\, dx\, dz = b^2 \int_0^c a\, dz = ab^2 c$$

On $S_3 : OADE\ \ x = 0\ \&\ \hat{n} = -\vec{i}$ and hence $\vec{f}.\hat{n} = -x^2 = 0$

$$\therefore \iint_{S_1} \vec{f}.\hat{n}\, ds = 0$$

On $S_4 : BCFG\ \ x = a\ \&\ \hat{n} = \vec{i}$ and hence $\vec{f}.\hat{n} = x^2 = a^2$

$$\therefore \iint_{S_4} \vec{f}.\hat{n}\, ds = \int_0^c \int_0^b a^2\, dy\, dz = a^2 \int_0^c b\, dz = a^2 bc$$

On $S_5 : OABG\ \ z = 0\ \&\ \hat{n} = -\vec{k}$ and hence $\vec{f}.\hat{n} = -z^2 = 0$

$$\therefore \iint_{S_1} \vec{f}.\hat{n}\, ds = 0$$

On $S_6 : CDEF\ \ z = c\ \&\ \hat{n} = \vec{k}$ and hence $\vec{f}.\hat{n} = z^2 = c^2$

$$\therefore \iint_{S_6} \vec{f}.\hat{n}\, ds = \int_0^a \int_0^b c^2\, dy\, dx = c^2 \int_0^a b\, dx = c^2 ba$$

$$\therefore \iint_{S} \vec{f}.\hat{n}\, ds = [a^2 cb + ab^2 c + abc^2] = abc[a+b+c] \quad .....(2)$$

From (1) & (2) Gauss Divergence Theorem is verified

Problem :-02

Verify Divergence theorem for $\vec{f} = 4xz\,\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$, taken over the cube bounded by the planes x=0, x=1, y=0, y=1, z=0 and z=1.

Solution:-

Gauss Divergence Theorem is $\iint_S \vec{f}.\hat{n}\, ds = \iiint_V \nabla.\vec{f}\, dV$

$$\nabla.\vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right) = 4z - 2y + y = 4z - y$$

$$\iiint_V \nabla.\vec{f}\ dV = \int_0^1\int_0^1\int_0^1 (4z - y)\ dz\,dy\,dx$$

$$= \int_0^1\int_0^1 [2z^2 - yz]_0^1\ dy\,dx$$

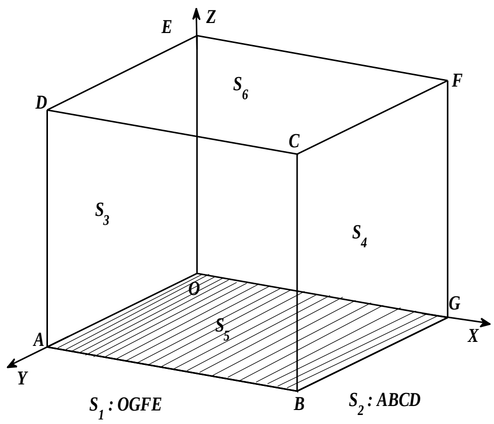
$$= \int_0^1\int_0^1 [2 - y]\ dy\,dx$$

$$= \int_0^1 \left[2y - \frac{y^2}{2}\right]_0^1\ dx$$

$$= \int_0^1 \frac{3}{2}\ dx$$

$$= \frac{3}{2} \text{.....(1)}$$

To evaluate $\iint_S \vec{f}.\hat{n}\ ds$ where $S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$



On $S_1 : OGFE\ y = 0\ \&\ \hat{n} = -\vec{j}$ and hence $\vec{f}.\hat{n} = y^2 = 0$

$$\therefore \iint_{S_1} \vec{f}.\hat{n}\ ds = 0$$

On $S_2 : ABCD\ y = 1\ \&\ \hat{n} = \vec{j}$ and hence $\vec{f}.\hat{n} = -y^2 = -1$

$$\therefore \iint_{S_2} \vec{f}.\hat{n}\ ds = \int_0^1\int_0^1 (-1)\ dx\ dz = -\int_0^1 dz = -1$$

On $S_3 : OADE\ x=0\ \&\ \hat{n}=-\vec{i}$ and hence $\vec{f}.\hat{n}=-4xz=0$

$$\therefore \iint_{S_1} \vec{f}.\hat{n}\ ds\ = 0$$

On $S_4 : BCFG\ x=1\ \&\ \hat{n}=\vec{i}$ and hence $\vec{f}.\hat{n}=4xz=4z$

$$\therefore \iint_{S_4} \vec{f}.\hat{n}\ ds\ = \int_0^1 \int_0^1 4z\ dy\ dz = 4\int_0^1 [zy]_0^1\ dz = 4\int_0^1 z\,dz = 4\left[\frac{z^2}{2}\right]_0^1 = 2$$

On $S_5 : OABG\ z=0\ \&\ \hat{n}=-\vec{k}$ and hence $\vec{f}.\hat{n}=-yz=0$

$$\therefore \iint_{S_1} \vec{f}.\hat{n}\ ds\ = 0$$

On $S_6 : CDEF\ z=1\ \&\ \hat{n}=\vec{k}$ and hence $\vec{f}.\hat{n}=yz=y$

$$\therefore \iint_{S_6} \vec{f}.\hat{n}\ ds\ = \int_0^1 \int_0^1 y\ dy\ dx = \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx = \frac{1}{2}\int_0^1 dx = \frac{1}{2}$$

$$\therefore \iint_{S} \vec{f}.\hat{n}\ ds\ = \left[-1+2+\frac{1}{2}\right] = \frac{3}{2} \quad .....(2)$$

From (1) & (2) Gauss Divergence Theorem is verified

Problem:-03

Verify divergence theorem for $\vec{F}\ =\ x^2\vec{i}\ + z\ \vec{j}\ +\ yz\ \vec{k}$ over the cube Formed by the planes $x = \pm1$, $y = \pm1$, $z = \pm1$.

Solution:-

Gauss Divergence Theorem is $\iint_S \vec{f}.\hat{n}\ ds\ = \iiint_V \nabla.\vec{f}\ dV$

$$\nabla.\vec{f}\ = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\left(x^2\vec{i}+z\vec{j}+yz\vec{k}\right) = 2x+y$$

$$\iiint_V \nabla.\vec{f}\ dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (2x+y)\ dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} [2x+y]\,[z]_{-1}^{1}\ dy\,dx$$

$$= 2\int_{-1}^{1}\int_{-1}^{1} [2x+y]\ dy\,dx$$

$$= 2\int_{-1}^{1} \left[2xy + \frac{y^2}{2}\right]_{-1}^{1} dx$$

$$= 2\int_{-1}^{1} \left[2x + \frac{1}{2} + 2x + \frac{1}{2}\right] dx$$

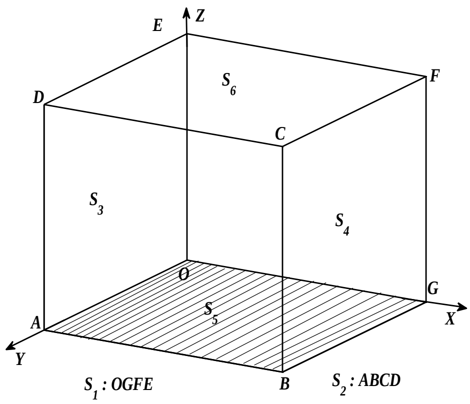
$$= 2\int_{-1}^{1} [4x + 1]\ dx$$

$$= 2\left[2x^2 + x\right]_{-1}^{1}$$

$$= 2[2 + 1 - 2 - 1]$$

$$= 0$$ .....(1)

To evaluate $\iint_S \vec{f}.\hat{n}\ ds$ where $S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$



On $S_1 : OGFE\ y = 0\ \&\ \hat{n} = -\vec{j}$ and hence $\vec{f}.\hat{n} = -z$

$$\therefore \iint_{S1} \vec{f}.\hat{n}\ ds = -\int_{-1}^{1}\int_{-1}^{1} z\ dx\ dz = -\int_{-1}^{1} [zx]_{-1}^{1}\ dz = -\int_{-1}^{1} [2z]\ dz = -[z^2]_{-1}^{1} = -[1-1] = 0$$

On $S_2 : ABCD\ y = 1\ \&\ \hat{n} = \vec{j}$ and hence $\vec{f}.\hat{n} = z$

$$\therefore \iint_{S2} \vec{f}.\hat{n}\ ds = \int_{-1}^{1}\int_{-1}^{1} z\ dx\ dz = \int_{-1}^{1} [zx]_{-1}^{1}\ dz = \int_{-1}^{1} [2z]\ dz = -[z^2]_{-1}^{1} = [1-1] = 0$$

On $S_3 : OADE\ x = -1\ \&\ \hat{n} = -\vec{i}$ and hence $\vec{f}.\hat{n} = -x^2 = -1$

$$\therefore \iint_{S3} \vec{f}.\hat{n}\ ds = \int_{-1}^{1}\int_{-1}^{1} (-1)\ dx\ dz = -\int_{-1}^{1} [2]\ dz = -2[2] = -4$$

On $S_4 : BCFG\ x = 1\ \&\ \hat{n} = \vec{i}$ and hence $\vec{f}.\hat{n} = x^2 = 1$

$$\therefore \iint_{S4} \vec{f}.\hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} 1\, dx\, dz = \int_{-1}^{1} [2]\, dz = 2[2] = 4$$

On $S_5 : OABG\ z=-1\ \&\ \hat{n}=-\vec{k}$ and hence $\vec{f}.\hat{n}=-yz=y$

$$\therefore \iint_{S5} \vec{f}.\hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} y\, dx\, dy = \int_{-1}^{1} [yx]_{-1}^{1}\, dy = \int_{-1}^{1} [2y]\, dy = [y^2]_{-1}^{1} = [1-1] = 0$$

On $S_6 : CDEF\ z=1\ \&\ \hat{n}=\vec{k}$ and hence $\vec{f}.\hat{n}=yz=y$

$$\therefore \iint_{S6} \vec{f}.\hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} y\, dx\, dy = \int_{-1}^{1} [yx]_{-1}^{1}\, dy = \int_{-1}^{1} [2y]\, dy = [y^2]_{-1}^{1} = [1-1] = 0$$

$$\therefore \iint_{S} \vec{f}.\hat{n}\, ds = [-4+4] = 0 \quad .....(2)$$

From (1) & (2) Gauss Divergence Theorem is verified

Problem:-04

Verify Gauss divergence theorem for $\vec{F} = (x^3 - yz)\vec{i} - 2x^2 y\vec{j} + 2\vec{k}$ over the cube bounded by x=0, y=0, z=0, x=a, y=a, z=a.

Solution:-

Gauss Divergence Theorem is $\iint_S \vec{f}.\hat{n}\, ds = \iiint_V \nabla.\vec{f}\, dV$

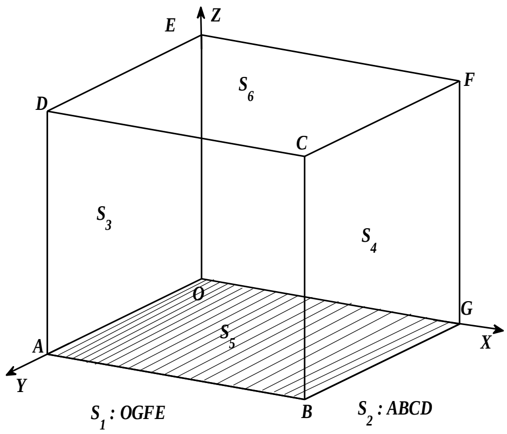
$$\nabla.\vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left((x^3-yz)\vec{i} - x^2y\vec{j} + 2\vec{k}\right) = 3x^2 - 2x^2 = x^2$$

$$\iiint_V \nabla.\vec{f}\, dV = \int_0^a\int_0^a\int_0^a (x^2)\, dz\, dy\, dx$$

$$= \int_0^a dz \int_0^a dy \int_0^a (x^2)\, dx$$

$$= a\,.\,a\,.\left[\frac{x^3}{3}\right]_0^a = a\,.\,a\,.\left[\frac{a^3}{3}\right] = \frac{a^5}{3} \quad ....(1)$$

To evaluate $\iint_S \vec{f}.\hat{n}\, ds$ where $S = S_1 + S_2 + S_3 + S_4 + S_5 + S_6$



On $S_1 : OGFE\ \ y = 0\ \&\ \hat{n} = -\vec{j}$ and hence $\vec{f}.\hat{n} = 2xy^2 = 0$

$$\therefore \iint_{S_1} \vec{f}.\hat{n}\ ds\ = 0$$

On $S_2 : ABCD\ \ y = a\ \&\ \hat{n} = \vec{j}$ and hence $\vec{f}.\hat{n} = -2x^2 y = -2x^2 a$

$$\therefore \iint_{S_2} \vec{f}.\hat{n}\ ds\ = -2a\int_0^a \int_0^a x^2\ dx\ dz = -2a\int_0^a \left[\frac{x^3}{3}\right]_0^a dz = -\frac{2a^4}{3}\int_0^a dz = -\frac{2a^5}{3}$$

On $S_3 : OADE\ \ x = 0\ \&\ \hat{n} = -\vec{i}$ and hence $\vec{f}.\hat{n} = -(x^3 - yz) = yz$

$$\therefore \iint_{S_3} \vec{f}.\hat{n}\ ds\ = \int_0^a \int_0^a yz\ dy\ dz = \int_0^a y\ dy \int_0^a z\ dz = \left[\frac{y^2}{2}\right]_0^a \left[\frac{z^2}{2}\right]_0^a = \frac{a^2}{2}\frac{a^2}{2} = \frac{a^4}{4}$$

On $S_4 : BCFG\ \ x = a\ \&\ \hat{n} = \vec{i}$ and hence $\vec{f}.\hat{n} = x^2 = (x^3 - yz) = a^3 - yz$

$\therefore$

$$\iint_{S_4} \vec{f}.\hat{n}\ ds\ = \int_0^a \int_0^a a^3 - yz\ dy\ dz = \int_0^a \int_0^a a^3\ dy\ dz - \int_0^a \int_0^a yz\ dy\ dz\ = a^3.a.a - \frac{a^4}{4}\ = a^5 - \frac{a^4}{4}$$

On $S_5 : OABG\ \ z = 0\ \&\ \hat{n} = -\vec{k}$ and hence $\vec{f}.\hat{n} = -2$

$$\therefore \iint_{S_5} \vec{f}.\hat{n}\ ds\ = \int_0^a \int_0^a (-2)\ dy\ dx = (-2)\int_0^a dy \int_0^a dx = -2a.a = -2a^2$$

On $S_6 : CDEF\ \ z = a\ \&\ \hat{n} = \vec{k}$ and hence $\vec{f}.\hat{n} = 2$

$$\therefore \iint_{S_6} \vec{f}.\hat{n}\ ds\ = \int_0^a \int_0^a (2)\ dy\ dx = (2)\int_0^a dy \int_0^a dx = 2a.a = 2a^2$$

$$\therefore \iint_{S} \vec{f}.\hat{n}\ ds\ = -2\frac{a^5}{3} + \frac{a^4}{4} + a^5 - \frac{a^4}{4} - 2a^2 + 2a^2\ = \frac{a^5}{3}$$ ....(2)

From (1) & (2) Gauss Divergence Theorem is verified

Problem:-05

Use Divergence theorem to evaluate $\iint_S f.nds$ where

$\vec{f} = 4x\,\vec{i} - 2\,\vec{j} + z^2\vec{k}$ and S is the surface bounding the region $x^2 + y^2 = 4$

z = 0 and z = 3.

Solution:-

Given $\vec{f} = 4x\,\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

$$\nabla.\vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}\right) = 4x - 4y + 2z$$

By Gauss Divergence Theorem $\iint_S \vec{f}.\hat{n}\,ds = \iiint_V \nabla.\vec{f}\,dV$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{3}(4-4y+2z)\,dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left(4z-4yz+z^3\right)_0^3\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)\,dy\,dx$$

$$= 21\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}dy\,dx - 12\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}y\,dy\,dx$$

= 21 (area of circle $x^2 + y^4 = 4$) – 0 { since y is odd}

= 21 (4π)

= 84π

STOKE'S THEOREM

The surface integral of the normal component of the curl of a vector function f over an open surface S is equal to the line integral of the tangential component of f around the closed curve C bounding S.

i.e. $\int_C \vec{f} \cdot d\,\vec{r} = \iint_S (\nabla \times \vec{f}) \cdot \hat{n}\, ds$

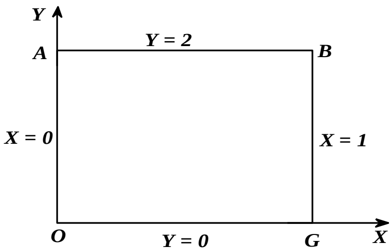
Problem:-01

Verify Stoke's theorem for $\vec{f}$ = xy $\vec{i}$ – 2yz $\vec{j}$ – xz $\vec{k}$ where S is the open surface of the rectangular parallopiped formed by the planes x =0, x = 1, y = 0, y = 2 and z = 3 above the XY plane.

Solution:-

Stoke's Theorem is $\int_C \vec{f} \cdot d\,\vec{r} = \iint_S (\nabla \times \vec{f}) \cdot \hat{n}\, ds$

$\vec{f}$ = xy $\vec{i}$ – 2yz $\vec{j}$ – xz $\vec{k}$



Here C is the boundary of the rectangle OGBAO, in the XOY plane bounded by the lines $x = 0, x = 1, y = 0, y = 2$.

r=x$\vec{i}$ +y$\vec{j}$ +z$\vec{k}$

=>dr= dx$\vec{i}$ +dy$\vec{j}$ +dz$\vec{k}$

Now $\vec{f}$ **.** dr **=** $\left[xy\vec{i} - 2yz\vec{j} - xz\vec{k}\right] \cdot \left(dx\vec{i} - dy\vec{j} - dz\vec{k}\right) = xy\,dx - 2yzdy - xzdz$

$\int_C$ $\vec{f}$ **.** dr **=** $\int_C \left[xy\vec{i} - 2yz\vec{j} - xz\vec{k}\right] \cdot \left(dx\vec{i} - dy\vec{j} - dz\vec{k}\right) = xy\,dx - 2yzdy - xzdz$

Along the line OG: $y=0,\ z=0$ and $dy=0,\ dz=0$. Also x varies from 0 to 1.

$\therefore \vec{f} \cdot dr = 0$ and hence $\int_{OG} \vec{f} \cdot dr = 0$

Along the line GB: $x=1,\ z=0$ and $dx=0,\ dz=0$. Also y varies from 0 to 2.

$\therefore \vec{f} \cdot dr = 0$ and hence $\int_{GB} \vec{f} \cdot dr = 0$

Along the line BA: $y=2,\ z=0$ and $dy=0,\ dz=0$. Also x varies from 1 to 0.

$\therefore \vec{f} \cdot dr = 2x\ dx$ and hence $\int_{BA} \vec{f} \cdot dr = \int_1^0 2x\ dx = \left[x^2\right]_1^0 = -1$

Along the line AO: $x=0,\ z=0$ and $dx=0,\ dz=0$. Also y varies from 2 to 0.

$\therefore \vec{f} \cdot dr = 0$ and hence $\int_{AO} \vec{f} \cdot dr = 0$

$\therefore \int_C \vec{f} \cdot dr = -1$ ....(1)

Also $\nabla\times\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -2yz & -xz \end{vmatrix} = \vec{i}(0+2y)-\vec{j}(-z-0)+\vec{k}(0-x) = 2y\vec{i}+z\vec{j}-x\vec{k}$

The surface S is the 5 surfaces of the parallopiped except $z=0$

To evaluate $\iint_S \left(\nabla\times\vec{f}\right).\hat{n}\ ds$ where $S = S_1+S_2+S_3+S_4+S_5$

On $S_1 : OGFE\ \ y=0\ \&\ \hat{n}=-\vec{j}$ and hence $\left(\nabla\times\vec{f}\right).\hat{n} = -z$

$\therefore \iint_{S_1} \left(\nabla\times\vec{f}\right).\hat{n}\ ds = -\int_0^1\int_0^3 z\ dz\,dx = -\int_0^1 dx\int_0^3 z\ dz = -(1)\left(\dfrac{z^2}{2}\right)_0^3 = -\dfrac{9}{2}$

On $S_2 : ABCD\ \ y=2\ \&\ \hat{n}=\vec{j}$ and hence $\left(\nabla\times\vec{f}\right).\hat{n} = z$

$$\therefore \iint_{S_2} \left(\nabla \times \vec{f}\right).\hat{n}\ ds = \int_0^1 \int_0^3 z\ dz\, dx = \int_0^1 dx \int_0^3 z\ dz = (1)\left(\frac{z^2}{2}\right)_0^3 = \frac{9}{2}$$

On $S_3 : OADE\ \ x = 0\ \&\ \hat{n} = -\vec{i}$ and hence $\left(\nabla \times \vec{f}\right).\hat{n} = -2y$

$$\therefore \iint_{S_3} \left(\nabla \times \vec{f}\right).\hat{n}\ ds = -\int_0^3 \int_0^2 2y\ dy\, dz = -2\int_0^3 dz \int_0^2 y\ dy = (-2\times 3)\left(\frac{y^2}{2}\right)_0^2 = -12$$

On $S_4 : BCFG\ \ x = 1\ \&\ \hat{n} = \vec{i}$ and hence $\left(\nabla \times \vec{f}\right).\hat{n} = 2y$

$$\therefore \iint_{S_4} \left(\nabla \times \vec{f}\right).\hat{n}\ ds = \int_0^3 \int_0^2 2y\ dy\, dz = 2\int_0^3 dz \int_0^2 y\ dy = (2\times 3)\left(\frac{y^2}{2}\right)_0^2 = 12$$

On $S_5 : CDEF\ \ z = 3\ \&\ \hat{n} = \vec{k}$ and hence $\left(\nabla \times \vec{f}\right).\hat{n} = -x$

$$\therefore \iint_{S_5} \vec{f}.\hat{n}\ ds = -\int_0^2 \int_0^1 x\ dx\ dy = -\int_0^2 dy \int_0^1 x\ dx = -2\left[\frac{x^2}{2}\right]_0^1 = -1$$

$$\therefore \iint_{S} \left(\nabla \times \vec{f}\right).\hat{n}\ ds = -12 + 12 - \frac{9}{2} + \frac{9}{2} - 1 = -1 \ldots\ldots\ldots(2)$$

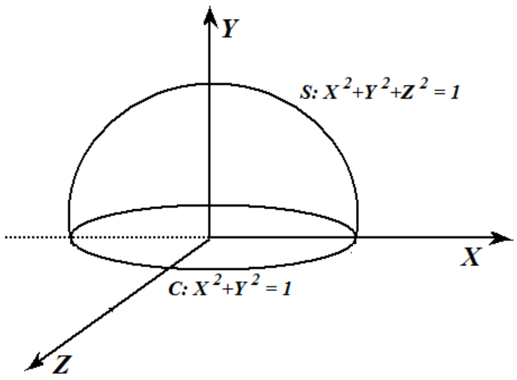
From (1) & (2), Stokes Theorem is verified

Problem:-02

Verify Stoke's theorem for vector field $\vec{f} = (2x - y)\,\vec{i} - yz^2\,\vec{j} - y^2 z\,\vec{k}$ over the upper half surface $x^2 + y^2 + z^2 = 1$, bounded by its projection on the xy plane.

Solution:-

Stoke's Theorem is $\int_C \vec{f}\,.\,d\,\vec{r} = \iint_S (\nabla \times \vec{f})\,.\,\hat{n}\ ds$



Here C is the boundary of the region R, which is the projection of the surface S on $z=0$ plane, namely the circle $x^2+y^2=1$.

Now

$\vec{f}\,.\,dr = \left[(2x-y)\vec{i}-yz^2\vec{j}-y^2z\vec{k}\right].\left(dx\vec{i}-dy\vec{j}-dz\vec{k}\right) = (2x-y)dx-yz^2dy-y^2zdz$

On C, $z=0$, $x^2+y^2=1$ **dz=0** and hence $\vec{f}\,.\,dr=(2x-y)dx$

$$\int_C \vec{f}.dr = \int_C (2x-y)\,dx$$

$$= \int_0^{2\pi} (2\cos\theta-\sin\theta)(-\sin\theta\,d\theta) \qquad \left|\begin{array}{l} put\; x=\cos\theta,\; y=\sin\theta \\ dx=-\sin\theta\,d\theta \end{array}\right.$$

$$= \int_0^{2\pi} \sin^2\theta\,d\theta - \int_0^{2\pi} \sin 2\theta\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1-\cos 2\theta)\;d\theta - \int_0^{2\pi} \sin 2\theta\,d\theta$$

$$= \frac{1}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{2\pi} - \left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= \pi - \left[-\frac{1}{2}+\frac{1}{2}\right] = \pi \;\ldots\ldots\ldots\; (1)$$

Also $\nabla\times\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \vec{i}(-2yz+2yz)-\vec{j}(0-0)+\vec{k}(0+1)=\vec{k}$

The unit normal vector to the surface $\phi: x^2+y^2+z^2=1$ is $\hat{n}=\dfrac{\nabla\phi}{|\nabla\phi|}$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2 - 1\right)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{1} = 2$$

$$\therefore \quad \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{2} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$ds = \frac{dx\,dy}{|\hat{n}.\vec{k}|} = \frac{dx\,dy}{|z|}$$

$$\iint_S \left(\nabla \times \vec{f}\right).\hat{n}\,ds = \iint_R z.\frac{dxdy}{|z|} = \iint_R dx\,dy = Area\,of\;\; R = \pi \text{ ....(2)}$$

From (1) and (2), Stoke’s Theorem is verified.

UNIT-2

ORDINARY DIFFERENTIAL EQUATION

Introduction:-

In mathematics, a *differential equation* is an equation that relates one or more functions and their derivatives. In applications, the functions generally represent *physical quantities*, the derivatives represent their *rates of change*, and the differential equation defines a *relationship between the two*. Such relations are common.

Mainly the study of differential equations consists of the study of their *solutions (the set of functions that satisfy each equation)*. Only the simplest differential equations are solvable by known methods, Often when a closed-form expression for the solutions is not available, in that case the solutions may be approximated numerically using computers.

Application:-

The differential equations play a prominent role in many disciplines including *engineering, physics, economics, and biology*.

The differential equations arise from many practical problems in oscillation of mechanical and electrical system, Bending of beams, Conduction of heat, velocity of chemical reactions etc.,

DIFFERENTIAL EQUATION (DE)

An equation involving derivatives of one or more *dependent variables* with respect to one or more *independent variables* is called a differential equation.

Example:-

$$\frac{d^4x}{dt^4}+\frac{d^2x}{dt^2}+\frac{dx}{dt}=e^t----(1)$$

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=0----(2)$$

$$\frac{dx}{dt}+\frac{dy}{dt}=10 \qquad -----(3)$$

$$2\frac{dx}{dt}-\frac{dy}{dt}=1, \quad 3\frac{dx}{dt}+\frac{dy}{dt}=0-----(4)$$

Note :-

(1) If there any changes in the value of y when x changes, then we say that y is a *dependent variable* and x is *independent* variable. ie y depends on x. Otherwise we say that y does not depend on x.

(2) If there is no change in the value one variable when other change then both are called *independent variables*.

(3) Equation (1), (2) and (3) are called *differential equations*, equation (4) is called *simultaneous differential equation*.

(4) In equation (1), 't' is independent variable and 'x' is dependent variable which depends on 't'.

(5) In equation (2), Independent variables are 'x', 'y', 'z' and dependent variable is 'u' which depends on x, y and z.

(6) In equation (3) & (4), Independent variables are 'x','y' and dependent variable is 't' which depends on x ,y.

ORDINARY DIFFERENTIAL EQUATION (ODE)

The equations having derivatives with respect to only *one independent variable* are called ODE.

Example:-

$$\frac{d^4x}{dt^4}+\frac{d^2x}{dt^2}+\frac{dx}{dt}=e^t----(1)$$

$$\frac{dx}{dt}+\frac{dy}{dt}=10 \qquad -----(2)$$

$$2\frac{dx}{dt}-\frac{dy}{dt}=1, \quad 3\frac{dx}{dt}+\frac{dy}{dt}=0-----(3)$$

$$dy=(x+\sin x)dx----(4)$$

Note:-

(1) The differential operator $\left(\frac{d}{dx},\frac{d^2}{dx^2},\frac{d^3}{dx^3}.......\right)$ in the differential equation are called *ordinary derivatives* .

(2) In other words, DE having *only one* independent variable is called *ODE*.

PARTIAL DIFFERENTIAL EQUATION

The equations having derivatives with respect to *at least two independent variable* are called ODE.

Example:-

$$\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=0.$$

$$\frac{\partial^2u}{\partial x^2}-5\frac{\partial u}{\partial y}=10$$

Note:-

(1) The differential operator $\left(\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}, \frac{\partial^3}{\partial x^3} \ldots\ldots\right)$ in the differential equation are called *partial derivatives* .

(2) In other words, DE's having *more than one* independent variables are called PDE.

ORDER OF A DIFFERENTIAL EQUATION

The order of a DE is the *highest-order derivative* that it involves.

Example:-

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \frac{dx}{dt} = e^t \qquad ----(1)$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 10 \qquad -----(2)$$

$$2\frac{dx}{dt} - \frac{dy}{dt} = 1, \quad 3\frac{dx}{dt} + \frac{dy}{dt} = 0 -----(3)$$

$$\frac{d^2x}{dt^2} + 10x = \cos t \qquad ----(4)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. ----(5)$$

$$\frac{\partial^3 u}{\partial x^3} - 5\frac{\partial u}{\partial y} = 10 \qquad ----(6)$$

Since the highest order derivative in ODE (1) is *4*, therefore the *order of the ODE is 4*.

Since the highest order derivative in ODE (2) is *1*, therefore the *order of the ODE is 1.*

Since the highest order derivative in ODE (3) is *1*, therefore the *order of the ODE is 1*.

Since the highest order derivative in ODE (4) is *2,* therefore the *order of the ODE is 2*.

Since the highest order derivative in PDE (5) is *2*, therefore the *order of the PDE is 2*

Since the highest order derivative in PDE (6) is *3*, therefore the *order of the PDE is 3.*

DEGREE OF A DIFFERENTIAL EQUATION

The degree of a DE is the *power of the highest order derivative*, after the equation has been made rational and integral in all of its derivatives.

EXAMPLE:-

$$\left(\frac{d^4x}{dt^4}\right)^1 + \left(\frac{d^2x}{dt^2}\right)^2 = e^t \qquad ----(1)$$

$$\left(\frac{dx}{dt}\right)^1 + \left(\frac{dy}{dt}\right)^1 = 10 \qquad -----(2)$$

$$2\left(\frac{dx}{dt}\right)^1 - \left(\frac{dy}{dt}\right)^1 = 1, \quad 3\left(\frac{dx}{dt}\right)^1 + \left(\frac{dy}{dt}\right)^1 = 0 -----(3)$$

$$\left(\frac{d^2x}{dt^2}\right)^2 + 10x = \cos t \qquad ----(4)$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 u}{\partial y^2}\right)^1 + \left(\frac{\partial^2 u}{\partial z^2}\right)^1 = 0. ----(5)$$

$$\left(\frac{\partial^3 u}{\partial x^3}\right) - 5\left(\frac{\partial u}{\partial y}\right)^3 = 10 \qquad ----(6)$$

Since the power of highest order derivative in ODE (1), (2) and (3) are *1*, therefore the *degree of the ODE's are 1.*

Since the power of highest order derivative in ODE (4) is *2*, therefore the *degree of the ODE's is 2.*

Since the power of highest order derivative in PDE (5) is *2*, therefore the *degree of the ODE's is 2.*

Since the power of highest order derivative in PDE (6) is *1*, therefore the *degree of the ODE's is 1*.

Since the power of highest order derivative in ODE (1) is *1,* therefore the *degree of the ODE is 1.*

EXACT DIFFERENTIAL EQUATION

A differential equation of the form $Mdx + Ndy = 0$, where M and N are function of x and y is called *exact differential equation* if and only if $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$.

Theorem

The Necessary and Sufficient condition for the DE $Mdx + Ndy = 0$ to be exact if $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$ or $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

METHOD OF SOLUTION OF EACT DIFFERENTIAL EQUATION

The solution of $Mdx + Ndy = 0$ is given by

$$\int_{\text{y constant}} Mdx + \int (\text{terms of N not containing x})dy = c$$

Problem:-01

Solve $2xydx + (x^2 + 3y^2)dy = 0.$

Solution:-

The given equation $2xydx + (x^2 + 3y^2)dy = 0.$ -----(1)

Equation (1) is of the form Mdx+Ndy=0 -----(2)

By comparing both the equation's (1), (2) left hand side, we get

Let $M = 2xy$ and $N = x^2 + 3y^2$

Differentiate M partially with respect to 'y' (Assuming 'x' constant), we get

$$\frac{\partial M}{\partial y} = 2x\frac{\partial (y)}{\partial y} = 2x.1 = 2x \qquad \text{-----(3)}$$

Differentiate N partially with respect to 'x' (Assuming 'y' constant), we get

$$\frac{\partial N}{\partial x} = \frac{\partial (x^2)}{\partial x} + 3y^2 \frac{\partial}{\partial x}(1) = 2x + 3y^2.0 = 2x \qquad \text{-----(4)}$$

$$\left(\because \frac{\partial (x^n)}{\partial x} = nx^{n-1} \quad \text{and} \quad \frac{\partial (c)}{\partial x} = 0 \right)$$

From the equations (3) and (4), we get

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Since the necessary and sufficient condition is satisfied, thus the given DE is exact.

Hence the solution of given exact DE is given by

$$\int\limits_{\text{y constant}} M\partial x + \int (\text{terms of N not containing x})\text{dy=c}$$

$$\int\limits_{\text{y constant}} 2xy\partial x + \int (3y^2)\text{dy=c}$$

$$2y \int\limits_{\text{y constant}} x\partial x + 3\int (y^2)\text{dy=c} \qquad \left(\because \int x^n \partial x = \frac{x^{n+1}}{n+1} \right)$$

$$2y\left(\frac{x^2}{2}\right) + 3\left(\frac{y^3}{3}\right) = c$$

$$yx^2 + y^3 = c$$

Which is required solution.

Problem -02

Solve $(2ye^{2x}+2x\cos y)dx+(e^{2x}-x^2\sin y)dy=0.$

Solution:-

The given DE is $(2ye^{2x}+2x\cos y)dx+(e^{2x}-x^2\sin y)dy=0.$ -----(1)

Equation (1) is of the form Mdx+Ndy=0 -----(2)

Comparing the above two equations, we get

$M=2ye^{2x}+2x\cos y, N=e^{2x}-x^2\sin y$

Differentiate M partially with respect to y (Assuming 'x' constant), we get

$$\frac{\partial M}{\partial y}=\frac{\partial}{\partial y}2ye^{2x}+\frac{\partial}{\partial y}2x\cos y$$

$$\frac{\partial M}{\partial y}=2e^{2x}\frac{\partial}{\partial y}y+2x\frac{\partial}{\partial y}\cos y$$

$$\frac{\partial M}{\partial y}=2e^{2x}.1+2x(-\sin y) \qquad \left(\because \frac{\partial \cos\ ax}{\partial x}=-a\sin ax\right)$$

$$\frac{\partial M}{\partial y}=2e^{2x}-2x\sin y \qquad \text{----(3)}$$

Differentiate N partially with respect to x (Assuming 'y' constant), we get

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}e^{2x}-\frac{\partial}{\partial x}x^2\sin y$$

$$\frac{\partial N}{\partial x}=\frac{\partial}{\partial x}e^{2x}-\sin y\frac{\partial}{\partial x}x^2$$

$$\frac{\partial N}{\partial x}=2e^{2x}-\sin y.2x \qquad \left(\because \frac{\partial e^{ax}}{\partial x}=ae^{ax} \quad \& \quad \frac{\partial x^n}{\partial x}=nx^{n-1}\right)$$

$$\frac{\partial N}{\partial x} = 2e^{2x} - 2x\sin y \qquad \text{----(4)}$$

By comparing the equations (3) and (4), we get

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Since the necessary and sufficient condition is satisfied, thus the given DE is exact.

Hence the solution of given exact DE is given by

$$\int_{\text{y constant}} M\,\partial x + \int (\text{terms of N not containing x})\text{dy=c}$$

$$\int_{\text{y constant}} (2ye^{2x} + 2x \ \cos y)\partial x + \int (0)\text{dy=c}$$

$$\int_{\text{y constant}} (2ye^{2x} + 2x\cos y)\partial x = c$$

$$2y \int_{\text{y constant}} e^{2x}\partial x + \cos y \int_{\text{y constant}} 2x\partial x + \int (0)\text{dy=c}$$

$$2y.\frac{e^{2x}}{2} + \cos y.\frac{x^2}{2} = c.$$

$$ye^{2x} + x^2 \cos y = c.$$

Which is the required solution.

Problem -03

Prove that the following equation is exact, find the solution $y' = \frac{3x^2 - 2xy}{x^2 - 2y}$

Solution:-

The given equation is $y' = \frac{3x^2 - 2xy}{x^2 - 2y}$ -----(1)

It can be rewritten as follows

$$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 - 2y}$$

$$(x^2 - 2y)dy = (3x^2 - 2xy)dx$$

$$(3x^2 - 2xy)dx + (2y - x^2)dy = 0 \qquad \text{-----(2)}$$

Equation (2) is of the form $Mdx + Ndy = 0.$-----(3)

Comparing the equations (2) and (3), we get

$$M = (3x^2 - 2xy) \qquad \& \quad N = (2y - x^2)$$

Differentiate M partially with respect to 'y' (Assuming 'x' constant),

we get

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}3x^2 - \frac{\partial}{\partial y}2xy$$

$$\frac{\partial M}{\partial y} = 3x^2\frac{\partial}{\partial y}(1) - 2x\frac{\partial}{\partial y}y$$

$$\frac{\partial M}{\partial y} = 3x^2.0 - 2x.1$$

$$\frac{\partial M}{\partial y} = 2x \qquad \text{----(4)}$$

Differentiate N partially with respect to 'x' (Assuming 'y' constant), we get

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x} 2y - \frac{\partial}{\partial x} x^2$$

$$\frac{\partial N}{\partial x} = 2y \frac{\partial}{\partial x}(1) - \frac{\partial}{\partial x} x^2$$

$$\frac{\partial N}{\partial x} = 2y.0 - 2x$$

$$\frac{\partial N}{\partial x} = 2x \qquad \text{----(5)}$$

By comparing the equations (4) and (5), we get

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Since the necessary and sufficient condition satisfied, therefore the given DE is exact.

Hence the solution of given exact DE is given by

$$\int_{\text{y constant}} M \partial x + \int (\text{terms of N not containing x})\text{dy=c}$$

$$\int_{\text{y constant}} (3x^2 - 2x\ y)\partial x + \int (2y)\text{dy=c}$$

$$3\int_{\text{y constant}} x^2\partial x - 2y\int_{\text{y constant}} x\,\partial x + 2\int y\text{dy=c}$$

$$3\frac{x^3}{3} - 2y.\frac{x^2}{2} + 2.\frac{y^2}{2} = c$$

$$x^3 - x^2y + y^2 = c.$$

Which is the required solution

EXERCISE

Solve the following equations

1. $y\ e^x dx + (2y + e^x)dy = 0$

2. $(x^2 - ay)dx = (ax - y^2)dy$

3. $(x^2 + y^2 - a^2)xdx + (x^2 - y^2 - b^2)ydy = 0$

4. $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

5. $(y^2e^{xy^2} + 4x^3)dx + (2xye^{xy^2} - 3y^2)dy = 0$ Ans. $e^{xy^2} + x^4 - y^3 = c$

6. $\left\{y\left(1 + \frac{1}{x}\right) + \cos y\right\}dx + (x + \log x - x\sin y)dy = 0$ Ans. (x+log x) y+x cos y=c

7. $(1 + 2xy\cos x^2 - 2xy)dx + (\sin x^2 - x^2)dy = 0$ Ans. $x + y\sin x^2 - yx^2 = c$

8. $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ Ans. $y\sin x + (\sin x + y)x = c$

9. $(2x^2+3y^2-7)xdx-(3x^2+2y^2-8)ydy=0$ Ans. $x^2 + y^2 - 3 = c(x^2 - y^2 - 1)^5$

10. $(3x^2+6xy^2)dx+(6x^2y+4y^3)dy=0$

LINEAR DES

A differential equation is said to *be linear*, if the following conditions are satisfied,

(i) **(a)** Derivative are of *degree one* in each term of DE

(b) Dependent variable appears with *degree one* in DE.

(ii) There *should not* be any term *containing the product of*

(a) Differential coefficient with dependent variable

(b) Differential coefficient with each other

(iii) Neither *differential coefficient* nor *dependent* variables are in *transcendental form*.

Note

(i) If any one of the condition violated, then the DE is *non linear*.

(ii) Transcendental form means that it involves trigonometric function like e^y, cos y, $e^{dy/dx}$, tan y, here y is dependent variable.

EXAMPLE:-

1. $\frac{dy}{dx} + y^{1/2} = \sin x$ is non linear (Condition *(i) (b) violated*)

(since the *degree* of the dependent variable y is *not equals one*)

2. y'''-6y'=5sinx is linear (No condition is violated)

3. $\frac{d^4y}{dx^4}+3\left(\frac{dy}{dx}\right)^2+y=x$ is non linear (Condition *(i) (a) violated*)

(since the *degree* of the y' is *not equals one*)

4. $\frac{d^4y}{dx^4}+3\sin x\frac{dy}{dx}=\cos x$ is linear (No Condition is *violated*)

5. $\frac{d^4y}{dx^4}.\frac{dy}{dx}+y=\cos x$ is non linear (Condition *(ii) (b) violated*)

(since the differential coefficient y'''' and y' are *multiplied each other*)

6. $y\frac{d^2y}{dx^2}+6\frac{dy}{dx}=0$ is non linear (Condition *(ii) (a) violated*)

(since the differential coefficient y'' and dependent variable y are *multiplied each other*)

7. $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+\cos y=\sin x$ is non linear (Condition *(iii) violated*)

(since the dependent variable y is in the *transcendental form*)

8. $\frac{d^2y}{dx^2}+e^y=\tan x$ is non linear (Condition *(iii) violated*)

(since the dependent variable y is in the *transcendental form*)

9. $\frac{d^2y}{dx^2}+e^x=\cos x$ is linear (No condition is *violated*)

10. $\frac{d^3y}{dx^3}+\frac{dy}{dx}+y^2=e^x\cos x$ is non linear (Condition *(i)(b) violated*)

(since the *degree* of the dependent variable y is *not equals to one*)

LEIBNITZ'S LINEAR EQUATION

The standard form of linear equation of first order commonly known as *Leibnitz's linear* equation

i.e $\frac{dy}{dx}+P(x)y=Q(x)$ is called *Leibnitz's linear* equation

It can be solved by using the following steps

Step :-01

Integrating factor: $e^{\int pdx}$

Step:-02

General Solution: $ye^{\int pdx}=\int Qe^{\int pdx}dx+c$.

BERNOULLI'S EQUATION

A first order DE that can be written in the form $\frac{dy}{dx}+P(x)y=Q(x)y^n$.

(n is real number) where P(x)& Q(x) are functions of x only (free from y) is called a Bernoulli's equation.

Note:-

(i) It is named after the Swiss Mathematician Jacob Bernoulli (1654-1705) who is known for his basic work in probability distribution theory.

(ii) It is clear that when n=0 or 1, the DE is linear.

(iii) It is clear that when n>1, the DE is non-linear.

(iv) It's solution is obtained by reducing the given Bernoulli's equation to Leibnitz's linear equation.

METHOD OF SOLUTION (WHEN n=0)

Consider the equation $\frac{dy}{dx}+P(x)y=Q(x)y^n$. where n=0.

i.e $\frac{dy}{dx}+P(x)y=Q(x)$

It becomes a Leibnitz's linear equation, so it can be solved by using the following steps

Step :-01

Integrating factor: $e^{\int pdx}$

Step:-02

General Solution: $ye^{\int pdx}=\int Qe^{\int pdx}dx+c$.

Problem:-01

Solve $\frac{dy}{dx}-y=\cos x$.

Solution:-

The given DE is $\frac{dy}{dx}-y=\cos x$.----(1)

It is of the form $\frac{dy}{dx}+P(x)y=Q(x)$----(2)

Comparing the equations (1) and (2), we have

$P=-1, \quad \& \quad Q=\cos x$

Integration factor

$$e^{\int Pdx} = e^{\int -dx} = e^{-\int dx} = e^{-x}$$

General solution

$$ye^{\int pdx} = \int Qe^{\int pdx}dx + c$$

$$ye^{-x} = \int \cos x e^{-x}dx + c \qquad \because \int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx + b\sin bx]$$

Here a=-1, b=1, substituting in the formula, we get

$$ye^{-x} = \frac{e^{-x}}{(-1)^2+(1)^2}[\sin x - \cos x] + c$$

$$ye^{-x} = \frac{e^{-x}}{2}[\sin x - \cos x] + c$$

Which is the required solution.

Problem:-02

Solve $\frac{1}{x}\frac{dy}{dx} + \frac{y}{x}\tan x = \cos x.$

Solution:-

The given DE is $\frac{1}{x}\frac{dy}{dx} + \frac{y}{x}\tan x = \cos x.$----(1)

Let us rewrite the above equation as follows

$$\frac{dy}{dx} + y\tan x = x\cos x. \qquad \text{(Multiply by x)}$$

It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$ ----(2)

Comparing the equations (1), (2), we get

$P = \tan x, \quad \& \quad Q = x\cos x$

Integration factor

$e^{\int Pdx} = e^{\int \tan x dx} = e^{\log \cos x^{-1}} = \sec x$

General solution

$ye^{\int pdx} = \int Qe^{\int pdx} dx + c$

$y\sec x = \int x\cos x \sec x dx + c$

$y\sec x = \int x dx + c$

$\therefore y\sec x = \frac{x^2}{2} + c$

Which is the required solution.

Problem :-03

Solve $\frac{dy}{dx} + y\cot x = \sin 2x.$

Solution :-

The given DE is $\frac{dy}{dx} + y\cot x = \sin 2x.$-----(1)

It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$----(2)

Comparing the equations (1), (2), we get

$P = \cot x, \quad \& \quad Q = \sin 2x$

Integration factor

$e^{\int Pdx} = e^{\int \cot x dx} = e^{\log \sin x} = \sin x$

General solution

$ye^{\int pdx} = \int Qe^{\int pdx} dx + c$

$y \sin x = \int \sin 2x \sin x dx + c$ $\quad \because \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

Here A=2x, B=x, substituting in the formula, we get

$y \sin x = \int \frac{1}{2}[\cos(2x-x) - \cos(2x+x)]dx + c$

$y \sin x = \frac{1}{2}\int[\cos(x) - \cos(3x)]dx + c$

$y \sin x = \frac{1}{2}[\frac{\sin x}{2} - \frac{\sin 3x}{3}] + c$ $\quad \left(\because \int \cos axdx = \frac{\sin ax}{a}\right)$

Which is required solution.

Problem:-04

Solve $\frac{dy}{dx} + y \cot x = 4x \cos ecx$, given that y=0 when x=$\frac{\pi}{2}$

Solution:-

The given DE is $\frac{dy}{dx} + y \cot x = 4x \cos ecx$ -----(1)

It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$ ----(2)

Given condition y(x)=0 when x=$\frac{\pi}{2}$-----(3) $[\because y = y(x)]$

Comparing the equations (1), (2), we get

$P = \cot x, \quad \& \quad Q = 4x\cos ecx$

Integration factor

$e^{\int Pdx} = e^{\int \cot xdx} = e^{\log\sin x} = \sin x$

General solution

$ye^{\int pdx} = \int Qe^{\int pdx}dx + c$

$y\sin x = \int 4x\cos ecx\sin xdx + c$

$y\sin x = \int 4xdx + c$

$y\sin x = 4\frac{x^2}{2} + c = 2x^2 + c$----(4)

To find the constant C

We use the given condition (3) in (4), we get

i.e sub x=$\frac{\pi}{2}$ and y=0 in equation (4), we get

$0.\sin(\frac{\pi}{2}) = 2(\frac{\pi^2}{4}) + c$

$0.1 = \frac{\pi^2}{2} + c$

$c = -\frac{\pi^2}{2}$

Substitute the value of c in equation (4), we get

$$\therefore\ y\sin x = 2x^2 - \frac{\pi^2}{2}.$$

Which is the required solution.

METHOD OF SOLUTION (WHEN n=1)

Consider the equation $\frac{dy}{dx} + P(x)y = Q(x)y^n$. when n=1.

i.e $\frac{dy}{dx} + P(x)y = Q(x)y.$

It can be solved using variable separable method.

i.e $\frac{dy}{dx} = -P(x)y + Q(x)y$

$$\frac{dy}{dx} = [Q(x) - P(x)]y$$

$\frac{dy}{y} = [Q(x) - P(x)]dx$ (Variables y and x are separated in left and right sides respectively)

Integrate on both sides, we get

$$\int\left(\frac{dy}{y}\right) = \int[Q(x) - P(x)]dx + c$$

After evaluation of integration on both sides, we get the required solution.

Problem:-01

Solve $\frac{dy}{dx}+xy=4y$

Solution:-

The given DE's is $\frac{dy}{dx}+xy=4y$ -----(1)

The equation (1) is of the form $\frac{dy}{dx}+P(x)y=Q(x)y.$-----(2)

Therefore we use variable separation method as follows

$$\frac{dy}{dx}=(4-x)y$$

$$\frac{dy}{y}=(4-x)dx$$

Integrate both sides, we get

$$\int \frac{dy}{y}=\int (4-x)dx$$

$$\log y=4x-\frac{x^2}{2}+c$$

$$\log y=4x-\frac{x^2}{2}+c$$

Which is the required solution.

Problem:-02

Solve $\frac{dy}{dx}+(x^2+2)y=(\tan x)y$

Solution:-

The given DE's is $\frac{dy}{dx}+(x^2+2)y=(\tan x)y$ -----(1)

The equation (1) is of the form $\frac{dy}{dx}+P(x)y=Q(x)y.$

Therefore we use variable separation method as follows

$$\frac{dy}{dx}=-(x^2+2)y+(\tan x)y$$

$$\frac{dy}{y}=[-x^2-2+\tan x]dx$$

Integrate both sides, we get

$$\int\frac{dy}{y}=\int[-x^2-2+\tan x]dx$$

$$\log y=-\frac{x^3}{3}-2x+\sec^2 x+c$$

Which is the required solution.

METHOD OF SOLUTION (WHEN n=2,3,4,....)

Consider the equation $\frac{dy}{dx} + P(x)y = Q(x)y^n.$ where n=2,3.

It is a non linear equation, but can be converted into linear equation as follows

$$\frac{dy}{dx} + P(x)y = Q(x)y^n. \text{ ----(1)}$$

Divide the equation by y^n

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^n}y = Q(x)\frac{1}{y^n}y^n.$$

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x).$$

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x).\text{----(2)}$$

Let $z=y^{1-n}$ (replacing the variable y into z)

$$\frac{dz}{dy} = (1-n)y^{1-n-1}$$

$$\frac{dz}{1-n} = y^{-n}dy$$

Update the values in equation(2), we have

$$\frac{y^{-n}dy}{dx} + P(x)y^{1-n} = Q(x).$$

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x).$$

Multiply 1-n on both sides , we get

$$\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x).$$

$$\frac{dz}{dx} + P_1(x)z = Q_1(x). \qquad \text{where } P_1(x) = (1-n)P(x) \;\; \& \; Q_1(x) = (1-n)Q(x)$$

This is the Leibnitz's linear equation in z.

This equation can be solved using following procedure

Integration factor

I.F= $e^{\int P_1(x)dx}$

General solution

$$ye^{\int P_1(x)dx} = \int Q_1(x)e^{\int P_1(x)dx}dx + c$$

Which gives the required solution.

Problem:-1

Solve $\frac{dy}{dx} + \frac{y}{x} = x^2 y^6$

Solution:-

The given DE is $\frac{dy}{dx} + \frac{y}{x} = x^2 y^6$ -----(1)

It is of the form $\frac{dy}{dx} + P(x)y = Q(x)y^n$. ----(2)

Rewrite equation (1) as follows

$$y^{-6}\frac{dy}{dx}+y^{-6}\frac{y}{x}=x^2y^{-6}y^6 \qquad \text{(Multply } y^{-6}\text{)}$$

$$y^{-6}\frac{dy}{dx}+\frac{1}{x}y^{-5}=x^2 \qquad \text{----(3)}$$

Let $z=y^{1-n}=y^{1-6}=y^{-5}$

Differentiate z with respect to y, we get

$$\frac{dz}{dy}=-5y^{-5-1}=-5y^{-6}$$

$$\frac{dz}{-5}=y^{-6}dy$$

Update the above values in equation (3), we get

$$y^{-6}\frac{dy}{dx}+\frac{1}{x}y^{-5}=x^2$$

$$\frac{1}{(-5)}\frac{dz}{dx}+\frac{1}{x}z=x^2$$

$$\frac{dz}{dx}-5\frac{1}{x}z=-5x^2 \qquad \text{-----(4)}$$

It is of the form $\frac{dz}{dx}+P(x)z=Q(x)$ ----(5)

Comparing the equations (4) and (5), we get

$P(x)=-5/x$ and $Q(x)=-5x^2$

Integrating factor

I.F= $e^{\int P(x)dx}=e^{\int\frac{-5}{x}dx}=e^{-5\log x}=e^{\log x^{-5}}=x^{-5}$

General solution

$$ze^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx + c$$

$$zx^{-5} = \int(-5x^2)x^{-5}dx + c$$

$$zx^{-5} = -5\int x^{-3}dx + c$$

$$zx^{-5} = -5\frac{x^{-3+1}}{-3+1} + c = -5\frac{x^{-2}}{-2} + c$$

$$zx^{-5} = \frac{5x^{-2}}{2} + c$$

$$z = \frac{5x^3}{2} + cx^5 \qquad \text{(Multiply } x^5\text{)}$$

Which is the general solution of equation (4).

The general solution of given equation (1) is obtained by replacing z by y.

Sub $z=y^{-5}$, we get

$$y^{-5} = \frac{5x^3}{2} + cx^5 \qquad \text{(Multiply } x^5\text{)}$$

Which is the required solution.

Problem:-2

Solve $xy(1+xy^2)\frac{dy}{dx} = 1$

Solution:-

The given DE is $xy(1+xy^2)\frac{dy}{dx} = 1$ -----(1)

Rewrite equation (1) as follows

$$\frac{dy}{dx} = \frac{1}{xy(1+xy^2)}$$

$$\frac{dx}{dy} = xy(1+xy^2)$$

$$\frac{dx}{dy} = xy + x^2y^3$$

$$\frac{dx}{dy} - xy = x^2y^3$$

$$\frac{dx}{dy} - yx = y^3x^2 \qquad \text{----(2)}$$

It is of the form $\frac{dx}{dy} + P(y)x = Q(y)x^n$. ----(3)

Multiply equation (2) by x^{-2}, we get

$$x^{-2}\frac{dx}{dy} - x^{-2}yx = x^{-2}y^3x^2$$

$$x^{-2}\frac{dx}{dy} - yx^{-1} = y^3 \text{-----(4)}$$

Let $z=x^{1-n}=x^{1-2}=x^{-1}$

Differentiate z with respect to x, we get

$$\frac{dz}{dx} = -x^{-1-1} = -x^{-2}$$

$$dz = -x^{-2}dx$$

Update the above values in equation (4), we get

$$-\frac{dz}{dy} - yz = y^3$$

$$\frac{dz}{dy} + yz = -y^3 \qquad \text{(Multiply by -1)} \qquad \text{-----(5)}$$

It is of the form $\frac{dz}{dy} + P(y)z = Q(y)$ ----(6)

Comparing the equations (5) and (6), we get

P(y)=y and Q(y)=-y^3

Integrating factor

I.F= $e^{\int P(y)dy} = e^{\int ydy} = e^{\frac{y^2}{2}}$

General solution

$$ze^{\int P(y)dy} = \int Q(y)e^{\int P(y)dy}dy + c$$

$$ze^{\frac{y^2}{2}} = \int (-y^3)e^{\frac{y^2}{2}}dx + c$$

$$ze^{\frac{y^2}{2}} = \int (-y^3)e^{\frac{y^2}{2}}dx + c$$

$$ze^{\frac{y^2}{2}} = -\int y^2 e^{\frac{y^2}{2}} ydy + c \qquad \text{----(7)}$$

Let $t=\frac{y^2}{2}$, $\frac{dt}{dy} = 2y$ => $\frac{dt}{2} = ydy$

Substitute the above values in the right side of equation (7), we get

$$ze^{\frac{y^2}{2}} = -\int (2t)e^t\left(\frac{dt}{2}\right) + c$$

$$ze^{\frac{y^2}{2}} = -\int te^t\, dt + c$$

$$ze^{\frac{y^2}{2}} = -\frac{1}{2}\left\{(t)\left(\frac{e^t}{1}\right) - (1)\left(\frac{e^t}{1}\right)\right\} + c \qquad \left(\text{chain rule of integration } \int udv = uv' - u'v'' + u''v''' - .....\right)$$

$$ze^{-\frac{y^2}{2}} = -\frac{1}{2}\left\{te^t - e^t\right\} + c$$

$$ze^{\frac{y^2}{2}} = \left\{-te^t + 2e^t\right\} + c$$

Sub $t=\frac{y^2}{2}$ in the above equation, we get

$$ze^{-\frac{y^2}{2}} = (2-t)e^{-t} + c$$

$$ze^{\frac{y^2}{2}} = (2-y^2)e^{\frac{y^2}{2}} + c$$

Which is the general solution of equation (5).

The general solution of given equation (1) is obtained by replacing z by x.

Sub $z=x^{-1}$, we get

$$x^{-1}e^{\frac{y^2}{2}} = (2-y^2)e^{\frac{y^2}{2}} + c$$

$$\frac{1}{x} = (2-y^2) + ce^{\frac{-y^2}{2}} \qquad (\text{Multiply on both sides } e^{\frac{-y^2}{2}})$$

Which is the required solution.

Problem:-3

Solve $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$

Solution:-

The given DE is $\frac{dy}{dx} + x\sin 2y = x^3 \cos^2 y$ -----(1)

Rewrite equation (1) as follows

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + x\frac{1}{\cos^2 y}\sin 2y = x^3 \qquad [\text{Divide the equation by } \cos^2 y]$$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + x\frac{1}{\cos^2 y}2\sin y\cos y = x^3 \qquad [\sin 2\theta = 2\sin\theta\cos\theta]$$

$$\sec^2 y\frac{dy}{dx} + x\frac{1}{\cos y}2\sin y = x^3$$

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3$$ --------(2)

Let us rewrite further

$$z = \tan y, \quad \frac{dz}{dy} = \sec^2 y \Rightarrow dz = \sec^2 y\,dy$$

Substitute the above in equation (2), we get

$$\frac{dz}{dx} + 2xz = x^3$$ -----(3)

It is of the form $\frac{dz}{dx} + P(x)z = Q(x)$ ----(4)

Comparing the equations (3) and (4), we get

$P(x)=2x$ and $Q(x)=x^3$

Integrating factor

I.F= $e^{\int P(x)dx} = e^{\int 2x dx} = e^{x^2}$

General solution

$$ze^{\int P(x)dx} = \int Q(x) e^{\int P(x)dx} dx + c$$

$$ze^{x^2} = \int x^3 e^{x^2} dx + c$$

$$ze^{x^2} = \int x^2 e^{x^2} x dx + c \quad \text{----(5)}$$

Let $t=x^2$, $\frac{dt}{dx} = 2x$ => $\frac{dt}{2} = xdx$

Substitute the above values in the right side of equation (6), we get

$$ze^{x^2} = \int (t) e^t \left(\frac{dt}{2}\right) + c$$

$$ze^{x^2} = \frac{1}{2} \int t e^t dt + c$$

$$ze^{x^2} = \frac{1}{2} \left\{ (t)\left(\frac{e^t}{1}\right) - (1)\left(\frac{e^t}{1}\right) \right\} + c \qquad \left(\text{chain rule of integration } \int u dv = uv' - u'v'' + u''v''' - .....\right)$$

$$ze^{x^2} = \frac{1}{2}\left\{ te^t - e^t \right\} + c$$

$$ze^{x^2} = \frac{1}{2}\left\{ t - 1 \right\} e^t + c$$

Sub $t=x^2$ in the above equation, we get

$$ze^{x^2} = \frac{1}{2}(t-1)e^t + c$$

$$ze^{x^2} = \frac{1}{2}(x^2-1)e^{x^2} + c$$

$$z = \frac{1}{2}(x^2-1) + ce^{-x^2} \qquad \text{(Multiply both sides by } e^{-x^2}\text{)}$$

Which is the general solution of equation (3).

The general solution of (1) is obtained by replacing z into y.

Sub z=tan y

$$\tan y = \frac{1}{2}(x^2-1) + ce^{-x^2} \qquad \text{(Multiply both sides by } e^{-x^2}\text{)}$$

Which is the required solution.

EXERCISE

Solve the following DE's

1. $\frac{dy}{dx}+\frac{y}{x}=x^2y^6$

2. $\frac{dz}{dx}+\frac{z}{x}\log z=\frac{z}{x}(\log z)^2$ Ans. $(\log z)^{-1}=1+cx$ [Use logz=t and 1/t=v]

3. $\frac{dy}{dx}=y\tan x-y^2\sec x$

4. $(x^3y^2+xy)dx=dy$

5. $2xy'=10x^3y^5+y$

6. $\frac{dy}{dx}=\frac{x^2+y^2}{2xy}$

7. $x(x-y)dy+y^2dy=0$

8. $\frac{dy}{dx}+\frac{\tan y}{1+x}=(1+x)e^x\sec y$

9. $e^y\left(\frac{dy}{dx}+1\right)=e^x$

10. $\tan y\frac{dy}{dx}+\tan x=\cos y\cos^2 x$

11. $6y'-2y=ty^4$ Ans. $\frac{1}{y^3}=\frac{1-t}{2}+ce^{-t}$

12. $y'-5y=-5ty^3$ Ans. $\frac{1}{y^2}=t-\frac{1}{10}+ce^{-10t}$

LINEAR D.E'S WITH VARIABLE COEFFICIENT

The general linear DE with variable coefficient of order 'n' is of the form $\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + ... + p_n y = X$ where $p_1, p_2, ..., p_n$ and X are *function of x only*.

LINEAR D.E'S WITH CONSTANT COEFFICIENTS

The general linear DE with constant coefficient of order 'n' is of the form $\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + ... + a_n y = X$, where $a_1, a_2, ..., a_n$ are *real constants*, and X is a *function of x only.*

This equation can also be written in symbolic form as follows,

$$(D^n y + a_1 D^{n-1} y + ... + a_n y) = f(x) \qquad \text{here } D = d/dx$$

$$(D^n + a_1 D^{n-1} + ... + a_n) y = f(x)$$

$$f(D) y = f(x) \text{ ----(1)}, \quad \text{where } f(D) = D^n + a_1 D^{n-1} + ... + a_n$$

The complete solution of equation (1) consists of two parts.

i) Complementary function.

ii) Particular integral.

ie., the *complete solution* is given by $y = y_c + y_p$. where y_c = C.F. , y_p = P.I.

Rules for find Complementary function

Write the *auxiliary equation* f(m)=0[replacing D by m in f(D)], and find its roots. Depending upon the nature of the roots we have the following cases

Case 1:

If all the roots $m_1, m_2, \ldots, m_n$ are real and different then the C.F. is,

$y_c = Ae^{m_1x} + Be^{m_2x} + Ce^{m_3x} + \ldots.$

Case 2:

If any two roots are equal say $m_1 = m_2 = m$ then the C.F. is,

$y_c = (Ax + B)e^{mx}$.

Case 3:

If any three roots are equal say $m_1 = m_2 = m_3 = m$, then the C.F. is,

$y_c = (Ax^2 + Bx + c)e^{mx}$.

Case 4:

If the roots are imaginary say $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$ then the C.F. is,

$y_c = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$.

Problem :-01

Find the C.F of $(D^2 - 6D + 13)y = 0.$

Solution:-

The given DE is $(D^2 - 6D + 13)y = 0$----(1)

Auxiliary equation is given by $m^2 - 6m + 13 = 0$ $[am^2 + bm + c = 0]$

$m = \frac{6 \pm \sqrt{36-4(13)}}{2}$ $m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$m = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm \sqrt{-16}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$

$C.F. = e^{3x}(A\cos 2x + B\sin 2x)$

Problem :-02

Find the C.F of $(D^2+1)y=0$

Solution:-

The given DE is $(D^2+1)y=0$ -----(1)

The auxiliary equation is $m^2+1=0$

$m^2 = -1$

$m = \sqrt{-1}$

$m = \pm i$ $[m = \alpha \pm \beta i, \quad \alpha=0, \ \beta=1]$

$C.F = e^{0x}[A\cos x + B\sin x]$ $[\text{C.F} = e^{\alpha x}(A\cos \beta x + B\sin \beta x)]$

$C.F = A\cos x + B\sin x$

Problem :-03

Find the C.F of $(D^3+D^2+4D+4)y=0$

Solution:-

The given DE is $(D^3+D^2+4D+4)y=0$ -----(1)

The auxiliary equation is $m^3 + m^2 + 4m + 4 = 0$

We use the following trial and error method to find the roots

	1	1	4	4
-1	0	-1	0	-4
	1	0	4	0

i.e $(m-(-1))(m^2+0m+4)=0$

i.e $(m+1)(m^2+4)=0$

$m+1=0$ or $m^2+4=0$

$m=-1$ or $m^2=-4$

$m=-1$ or $m=\pm 2i$

$C.F=C_1e^{-x}+e^{0x}[C_2\cos 2x+C_3\sin 2x]$

Note:-

The trial and error method is applicable only if at least one root of the equation is real.

Problem :-04

Find the C.F of $(D^4 - 4D^2 + 4)y = 0$

Solution:-

The given DE is $(D^4 - 4D^2 + 4)y = 0$ -----(1)

The auxiliary equation is $m^4 - 4m^2 + 4 = 0$

For this problem the trial and error method not applicable (Since all the roots are complex), So we find the root by rearranging the equation as follows

$(m^2)^2 - 4m^2 + 4 = 0$

$(n)^2 - 4n + 4 = 0$, where $n = m^2$

$n^2 - 4n + 4 = 0$ [$a n^2 + bn + c = 0$]

$$n = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)} \qquad \left(n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \right)$$

$$n = \frac{4 \pm \sqrt{0}}{2} = \frac{4 \pm 0}{2} = 2 \pm 0$$

$$n = 2 + 0, 2 - 0$$

$$n = 2, 2$$

$$m^2 = 2, \qquad m^2 = 2, \qquad [m^2 = n]$$

$$m = \pm\sqrt{2}, \qquad m = \pm\sqrt{2}$$

Hence the four roots are $m = \sqrt{2}, -\sqrt{2}, \sqrt{2}, -\sqrt{2}$

C.F = $(C_1 + C_2 x)e^{\sqrt{2}x} + (C_3 + C_4 x)e^{-\sqrt{2}x}$

Problem :-05

Find the C.F of $(D^4 + 4)y = 0$

Solution:-

The given DE is $(D^4+4)y=0$ -----(1)

The auxiliary equation is $m^4+4=0$

For this problem the trial and error method not applicable (Since all the roots are complex), So we find the root by rearranging the equation as follows

$(m^2)^2+4=0$

$(n)^2+4=0$, where $n=m^2$

$n^2+4n+4-4n=0$

$(n+2)^2-4n=0$ $[(a+b)^2=a^2+2ab+b^2]$

$(m^2+2)^2-2^2m^2=0$

$(m^2+2)^2-(2m)^2=0$ $[a^2-b^2=(a-b)(a+b)]$

$(m^2+2-2m)(m^2+2+2m)=0$

$m^2-2m+2=0$ or $m^2+2m+2=0$

To find the roots of $m^2-2m+2=0$

$$m=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)} \qquad \left(m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$$

$$m=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm\sqrt{-4}}{2}=\frac{2\pm 2i}{2}$$

$m=1\pm i$

To find the roots of $m^2+2m+2=0$

$$m=\frac{-(2)\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)} \qquad \left(m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$$

$$m=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm\sqrt{-4}}{2}=\frac{-2\pm 2i}{2}$$

$$m=-1\pm i$$

Hence the four roots are

$m=1\pm i,\quad -1\pm i,$ [For $1\pm i$, α=1, β=1 & -1$\pm i$, α=-1, β=1]

C.F = $e^{-x}[C_1\cos x+C_2\sin x]+e^{+x}[C_3\cos x+C_4\sin x]$

EXERCISE

Find the C.F of the following DE's

1. $(D^2-2)^2 y=0.$

2. $(D^3-1)y=0.$

3. $\frac{d^2x}{dt^2}+5\frac{dx}{dt}+6x=0$

4. $\frac{d^2x}{dt^2}+6\frac{dx}{dt}+9x=0$

5. y''+3y'+2y=0

6. y''+y'+y=0

7. $(D^2+4)^2y=0$

8. $(D-3)^4y=0$

9. $(D+8)^4y=0$

10. 4y'''+4y''+y'=0

11. $(D^3+1)y=0$

12. $(D^2+1)^2(D-1)y=0$

13. $(4D^4-8D^3-7D^2+11D+6)y=0$

14. $(D^3-3D^2+3D-1)\, y=0$

15. $4y'''+4y''+y'=0$

16. $(D^4+8D^2+16)y=0$

Rules for finding particular Integral

Consider the general linear DE with constant coefficient of order 'n'

$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + ... + a_n y = X$, where $a_1, a_2, ..., a_n$ are *real constants*, and X is a *function of x only.*

This equation can also be written in symbolic form as follows,

f(D)$y = X$ ----(1), where f(D)=$D^n + a_1 D^{n-1} + ... + a_n$

The particular integral is obtained as follows

P.I= $\frac{1}{f(D)} X = \frac{1}{D^n + a_1 D^{n-1} + ... + a_n} X$

Note:-

Formulae $\frac{1}{D} X = \int X dx$ and $\frac{1}{D^2} X = \int \int X dx dx$ etc.,

TYPE: 01

If $X = e^{ax}$, then the P.I. is

P.I. = $\frac{1}{\phi(D)} e^{ax}$ [Factorize the denominator ϕ(D) in to linear factors]

P.I = $\frac{1}{\phi(a)} e^{ax}, provied \phi(a) \neq 0.$

If $\phi(a) = 0.$ then multiply by x and differentiate the denominator with respect D, and replace D by a.

P.I = $x\frac{1}{\phi'(a)}e^{ax}, provied\,\phi'(a) \neq 0.$

Continue this procedure until we get the required solution.

Problem:-01

Solve $\frac{d^2y}{dx^2} - 4y = 6e^{5x}.$

Solution:-

The given DE is $\frac{d^2y}{dx^2} - 4y = 6e^{5x}.$-----(1)

Let write the symbolic form of equation (1) as follows

$D^2y - 4y = 6e^{5x}.$ where $D = \frac{d}{dx}.$

$(D^2 - 4)y = 6e^{5x}$ -----(2)

The complete solution of equation (2) is given by

y=C.F+P.I -----(3)

To find C.F

Auxiliary equation is $m^2 - 4 = 0$

$m^2 = 4$

$m = \pm\sqrt{4}$

$m = \pm 2$

$C.F = Ae^{2x} + Be^{-2x}$

To find P.I

$$P.I. = \frac{1}{f(D)} X = \frac{1}{D^2 - 4} 6e^{5x}$$

Let us write the denominator in linear factor form $D^2 - 2^2 = (D-2)(D+2)$

$$P.I. = 6\frac{1}{(D-2)(D+2)} e^{5x}$$

$$P.I. = 6\frac{1}{(5-2)(5+2)} e^{5x} \qquad \text{[case I, replace D by 5]}$$

$$P.I. = \frac{6}{21} e^{5x} = \frac{2}{7} e^{5x}$$

Substituting the values of C.F and P.I in equation (3), we get

y= $Ae^{2x} + Be^{-2x} + \frac{2}{7} e^{5x}$

Which is the required solution.

Problem:-02

Find the particular integral of $y'' - 3y' + 2y = e^x - e^{2x}$.

Solution:-

The given equation is $y'' - 3y' + 2y = e^x - e^{-2x}$. -----(1)

Let us rewrite the equation (1) in symbolic form as follows

$D^2 y - 3Dy + 2y = e^x - e^{-2x}$. where $D = \frac{d}{dx}$.

$(D^2 - 3D + 2)y = e^x - e^{-2x}$. -----(2)

To find P.I

$$P.I. = \frac{1}{f(D)}X = \frac{1}{D^2 - 3D + 2}(e^x - e^{-2x})$$

$$P.I. = \frac{1}{D^2 - 3D + 2}e^x - \frac{1}{D^2 - 3D + 2}e^{-2x} \qquad \text{-----(3)}$$

Let write the denominator in linear factorize form

D^2-3D+2=D^2-D-2D+2

D^2-3D+2=D(D-1)-2(D-1)

D^2-3D+2=(D-2)(D-1)

Now

$$\frac{1}{(D-1)(D-2)}e^x = \frac{1}{(1-1)(1-2)}e^x$$ [Case I, replace D by 1, denominator is zero]

$$\frac{1}{D^2 - 3D + 2}e^x = \frac{1}{(D-1)(-1)}e^x$$ [Case I, replace D by 1, denominator is zero]

$$\frac{1}{D^2 - 3D + 2}e^x = -\frac{1}{(D-1)}e^x$$ [Case I, replace D by 1, denominator is zero]

If denominator becomes zero, multiply numerator by x and differentiate denominator with respect to D and then replace D by 1 again.

$$\frac{1}{D^2 - 3D + 2}e^x = -x\frac{1}{1-0}e^x \qquad [\because \frac{d}{dD}(D-1) = 1 - 0 = 1]$$

$$\frac{1}{D^2 - 3D + 2}e^x = -xe^x \qquad \text{-----(4)}$$

$$\frac{1}{(D-1)(D-2)}e^{-2x} = \frac{1}{(-2-1)(-2-2)}e^{-2x}$$ [Case I, replace D by -2, denominator is non zero]

$$\frac{1}{D^2-3D+2}e^{-2x} = \frac{1}{12}e^{-2x}$$ ----(5)

Substitute the equation (4), (5) in (3), we get

P.I $= -xe^{x} - \frac{1}{12}e^{-2x}$

Which is the required answer.

Problem -03

Find the particular integral of $(D-1)^2 y = \sinh 2x.$

Solution:

The given DE is $(D-1)^2 y = \sinh 2x.$ -----(1), it is in symbolic form only.

To find P.I

P.I $= \frac{1}{f(D)}X = \frac{1}{(D-1)^2}\sinh 2x$

Here the denominator is already in linear factor form

P.I $= \frac{1}{(D-1)^2}\left(\frac{e^{2x}-e^{-2x}}{2}\right)$ $\quad [\sinh 2x = (\frac{e^{2x}-e^{-2x}}{2})]$

P.I $= \frac{1}{2}\frac{1}{(D-1)^2}e^{2x} - \frac{1}{2}\frac{1}{(D-1)^2}e^{-2x}$ -----(2)

Now

$$\frac{1}{(D-1)^2}e^{2x} = \frac{1}{(2-1)^2}e^{2x}$$ [Case I, replace D by 2, denominator non zero]

$$\frac{1}{(D-1)^2}e^{2x} = e^{2x}$$ -----(3)

$$\frac{1}{(D-1)^2}e^{-2x} = \frac{1}{(-2-1)^2}e^{-2x}$$ [Case I, replace D by -2, denominator non zero]

$$\frac{1}{(D-1)^2}e^{-2x} = \frac{e^{-2x}}{9}$$ -----(4)

Substitute the equations (3), (4) in (2), we get

P.I= $\frac{1}{2}e^{2x} - \frac{1}{2}\frac{e^{-2x}}{9}$

P.I= $\frac{1}{2}\left(e^{2x} - \frac{e^{-2x}}{9}\right)$

Problem -04

Find the particular integral of $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x.$

Solution:

The given DE is $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x.$-----(1)

It is already in symbolic form.

P.I= $\frac{1}{f(D)}X = \frac{1}{(D+2)(D-1)^2}\left(e^{-2x} + 2\sinh x.\right)$

Here the denominator is already in linear factor form

P.I= $\frac{1}{(D+2)(D-1)^2}e^{-2x}+\frac{1}{(D+2)(D-1)^2}2\sinh x.$

P.I= $\frac{1}{(D+2)(D-1)^2}e^{-2x}+\frac{1}{(D+2)(D-1)^2}2.\left(\frac{e^x-e^{-x}}{2}\right).$ [sinhx= $\frac{e^x-e^{-x}}{2}$]

P.I= $\frac{1}{(D+2)(D-1)^2}e^{-2x}+\frac{1}{(D+2)(D-1)^2}(e^x-e^{-x})$

P.I= $\frac{1}{(D+2)(D-1)^2}e^{-2x}+\frac{1}{(D+2)(D-1)^2}e^x-\frac{1}{(D+2)(D-1)^2}e^{-x}$ -----(2)

Now

$$\frac{1}{(D+2)(D-1)^2}e^{-2x}=\frac{1}{(D+2)(D-1)^2}e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2}e^{-2x}=\frac{1}{(-2+2)(-2-1)^2}e^{-2x}$$ [Case I, replace D by -2, denominator is zero]

$$\frac{1}{(D+2)(D-1)^2}e^{-2x}=\frac{1}{9(D+2)}e^{-2x}$$ [Case I, replace D by -2, denominator is zero]

If denominator becomes zero, multiply by x and differentiate denominator with respect to D and then replace D by -2 again.

$$\frac{1}{(D+2)(D-1)^2}e^{-2x}=\frac{1}{9}x\frac{1}{(1+0)}e^{-2x}$$ [$\because \frac{d}{dD}(D+2)=1+0=1$]

$$\frac{1}{(D+2)(D-1)^2}e^{-2x}=\frac{xe^{-2x}}{9}$$ -----(3)

$$\frac{1}{(D+2)(D-1)^2}e^{x}=\frac{1}{(1+2)(1-1)^2}e^{x}$$ [Case I, replace D by 1, denominator is zero]

$$\frac{1}{(D+2)(D-1)^2}e^x = \frac{1}{3(D-1)^2}e^x$$ [Case I, replace D by 1, denominator is zero]

If denominator becomes zero, multiply by x and differentiate denominator with respect to D and then replace D by 1 again.

$$\frac{1}{(D+2)(D-1)^2}e^x = \frac{1}{3}x\frac{1}{(2D-2)}e^x = \frac{x}{6}\frac{1}{(D-1)}e^x$$

$$[\ \because \frac{d}{dD}(D-1)^2 = \frac{d}{dD}(D^2-2D+1) = 2D-2+0 = 2D-2]$$

$$\frac{1}{(D+2)(D-1)^2}e^x = \frac{x}{6}\frac{1}{(1-1)}e^x$$ [Case I, replace D by 1 again, denominator is zero]

If denominator becomes zero, multiply by x and differentiate denominator with respect to D and then replace D by 1 again.

$$\frac{1}{(D+2)(D-1)^2}e^x = \frac{x}{6}.x.\frac{1}{(1-0)}e^x \qquad [\because \frac{d}{dD}(D-1) = 1-0 = 1]$$

$$\frac{1}{(D+2)(D-1)^2}e^x = \frac{x^2e^x}{6}. \qquad \text{-----(4)}$$

$$\frac{1}{(D+2)(D-1)^2}e^{-x} = \frac{1}{(-1+2)(-1-1)^2}e^{-x}$$

[Case I, replace D by -1 again, denominator is non zero]

$$\frac{1}{(D+2)(D-1)^2}e^{-x} = \frac{1}{(1)(-2)^2}e^{-x} = \frac{1}{4}e^{-x} \qquad \text{-----(5)}$$

Substitute the equations (3), (4) and (5) in (2), we get

P.I= $\frac{xe^{-2x}}{9}+\frac{x^2e^x}{6}.+\frac{1}{4}e^{-x}$

Which is the required answer.

Problem -04

Solve $(D\ -1)^3 y = e^x$.

Solution:

The given DE is $(D\ -1)^3 y = e^x$. -----(1)

It is already in symbolic form

Its complete solution is given by y=C.F+P.I------(2)

To find C.F

The auxiliary equation is $(m-1)^3=0$

(m-1)(m-1)(m-1)=0

m-1=0 or m-1=0 or m-1=0

m=1,1,1 (Thrice)

C.F=$(A+Bx+cx^2)e^x$ ------(3)

To find P.I

P.I= $\frac{1}{f(D)}=\frac{1}{(D-1)^3}e^x$ [Case I, replace D by 1, denominator is zero]

P.I= $\frac{1}{(1-1)^3}e^x$

If denominator becomes zero, multiply by x and differentiate denominator with respect to D and then replace D by 1 again.

P.I= $x.\frac{1}{3(D-1)^2}e^x$ $\quad [\because \frac{d}{dD}(D-1)^3 = 3(D-1)^{3-1}(1-0) = 3(D-1)^2]$

P.I= $\frac{x}{3}\frac{1}{(1-1)^2}e^x$ $\quad$ [Case I, replace D by 1 again, denominator is zero]

If denominator becomes zero, multiply by x and differentiate denominator with respect to D and then replace D by 1 again.

P.I= $\frac{x}{3}.x.\frac{1}{2(D-1)}e^x$ $\quad [\because \frac{d}{dD}(D-1)^2 = 2(D-1)^{2-1}(1-0) = 2(D-1)]$

P.I= $\frac{x^2}{6}\frac{1}{(1-1)}e^x$ $\quad$ [Case I, replace D by 1 again, denominator is zero]

If denominator becomes zero, multiply by x and differentiate denominator with respect to D and then replace D by 1 again.

P.I= $\frac{x^2}{6}.x.\frac{1}{(1)}e^x$ $\quad [\because \frac{d}{dD}(D-1) = 1-0 = 1]$

P.I= $\frac{x^3e^x}{6}$ ------(4)

Substitute the equation (3) and (4) in (2), we get

$y=(A+Bx+cx^2)e^x+\frac{x^3e^x}{6}$

Which is the required answer.

EXERCISE

Solve the following DE's

1. $(D^2-2D+1)y=\cosh x.$.

2. $(D^2-4)y=10e^{3x}-3e^{-5x}.$.

3. $y''+y'+y=(1-e^x)^2$.

4. $(D-2)^2y=8e^{2x}$.

5. $y''-6y'+9y=6e^{3x}-7e^{2x}$.

6. $y'''-y'=2e^x$.

7. $y''+2y'+y=e^{3x}$.

8. $y''-y=e^x$.

9. $y''-6y'+25y=e^{2x}$.

Type 2:

If $X = \sin ax \quad (or) \quad \cos ax,$, then the P.I. is

P.I. = $\frac{1}{f(D)} \sin ax \quad (or) \quad \cos ax$

Replace D² (*only even powers of D*) by – a² in $f(D)$ provided $f(-a^2) \neq 0$.

Note:-

1. In this type 2, we can replace only the following value D^2 , D^4, D^6. .,.... etc. i.e *Even powers* of D can only be replaced by a number, but *not odd D*. If there are any odd power of D in the denominator make it as even power by suitable multiplication of linear factor in denominator and numerator, then substitute the even powers of D etc.,

2. In case of denominator zero, multiply x in the numerator and differentiate the function f(D) in the denominator with respect to D. Again substitute the values of event powers of D.

Problem:01

Solve $(D^2+3D+2)y=\sin 3x\cos 2x$

Solution:-

The given DE is $(D^2+3D+2)y=\sin 3x\cos 2x$-----(1)

It is in symbolic form.

The complete solution of (1) is given by y=C.F+P.I-----(2)

To find C.F

Auxiliary equation is $m^2+3m+2=0$

m=$\dfrac{-(3)\pm\sqrt{(3)^2-4(1)(2)}}{2(1)}$ $\left(m=\dfrac{-b\pm\sqrt{b^{\partial 2}-4ac}}{2a}\right)$

m=$\dfrac{-3\pm\sqrt{9-8}}{2}=\dfrac{-3\pm1}{2}$

m=$\dfrac{-3+1}{2}\quad or\quad\dfrac{-3-1}{2}$

m=$-1\quad or\quad -2$

$C.F.=Ae^{-x}+Be^{-2x}$

To Find P.I

P.I=$\dfrac{1}{f(D)}X=\dfrac{1}{D^2+3D+2}\sin 3x\cos 2x$ $\left(\because \sin A\cos B=\dfrac{1}{2}[\sin(A+B)+sin(A-B)]\right)$

P.I$=\dfrac{1}{(D^2+3D+2)}\dfrac{1}{2}[\sin(3x+2x)+\sin(3x-2x)]$

$$P.I = \frac{1}{2}\frac{1}{D^2+3D+2} sin5x + \frac{1}{2}\frac{1}{D^2+3D+2}\sin x$$

$$P.I = \frac{1}{2}[\frac{1}{3D-23}\sin 5x + \frac{1}{3D+1}\sin x]$$

[CASE-II $D^2 = -(5)^2 = -25$, Second $D^2 = -(1)^2 = -1$]

$$P.I = \frac{1}{2}[\frac{3D+23}{9D^2-529}\sin 5x + \frac{3D-1}{9D^2-1}\sin x]$$

$$= \frac{1}{2}[\frac{3D+23}{-754}\sin 5x + \frac{3D-1}{-10}\sin x]$$

$$= \frac{1}{2}[\frac{3D\sin 5x + 23\sin 5x}{-754} + \frac{3D\sin x - \sin x}{-10}]$$

$$= \frac{1}{2}[\frac{15\cos 5x + 23\sin 5x}{-754} + \frac{3\cos x - \sin x}{-10}]$$

$$\therefore y = C.F. + P.I.$$

Problem:-02

$2. Solve: (D^2-4D-5)y = \cos x + e^{-x}$

$so \ln:$

$A.E. m^2 - 4m - 5 = 0$

$m = -1, 5.$

$C.F. = Ae^{5x} + Be^{-x}$

$$P.I. = \frac{1}{D^2-4D-5}(\cos x + e^{-x})$$

$$= \frac{1}{(D^2-4D-5)}\cos x + \frac{1}{D^2-4D-5}e^{-x}$$

$$= \frac{-(4D-6)}{16D^2-36}\cos x + \frac{x}{2D-4}e^{-x}$$

$$= \frac{2sinx - 3\cos x}{-26} - \frac{x}{6}e^{-x}$$

$\therefore y = C.F. + P.I.$

$$= \frac{1}{D^2+1}\sin x \quad D^2 = -(1)^2 = -1$$

$$= \frac{1}{-1+1}\sin x$$

$$= x\frac{1}{2D+0}\sin x \quad D^2 = -1$$

$$= \frac{1}{2}x\frac{1}{D}\frac{D}{D}\sin x \quad D^2 = -1$$

$$= \frac{1}{2}x\frac{D}{(-1)}\sin x$$

$$= \frac{1}{2}x\frac{\cos x}{-1}$$

$$= \frac{-x\cos x}{2}$$

EXERCISE

1. $Solve:(D^3+2D^2+D)y = \sin x + e^{-2x}.$

Type 3

If $f(x) = a_0x^n + a_1x^{n-1} + ... + a_n, where a_0x^n + a_1x^{n-1} + ... + a_n$ is a pure algebraic function then , P.I. = $\frac{1}{\phi(D)}(a_0x^n + a_1x^{n-1} + ... + a_n)$

$$= \phi(D)^{-1}(a_0x^n + a_1x^{n-1} + ... + a_n).$$

Expand $\phi(D)^{-1}$ by using Binomial theorem in ascending powers of D and then operator on $a_0x^n + a_1x^{n-1} + ... + a_n$.

Problem:-01

$1. Solve: (D^3 - 3D^2 - 6D + 8)y = x.$

$so\ln:$

$A.E., m^3 - 3m^2 - 6m + 8 = 0$

$\therefore m = 1, -2, 4$

$\therefore C.F. = Ae^x + Be^{-2x} + Ce^{4x}$

$$P.I. = \frac{1}{(D^3 - 3D^2 - 6D + 8)} x$$

$$= \frac{1}{8[1 + (\frac{D^3 - 3D^2 - 6D}{8})]} x$$

$$= \frac{1}{8}[1 + (\frac{D^3 - 3D^2 - 6D}{8})]^{-1}(x)$$

$$= \frac{1}{8}[1 - (\frac{D^3 - 3D^2 - 6D}{8})](x)$$

$$= \frac{1}{8}[x - (-\frac{6}{8})]$$

$$= \frac{1}{8}(x + \frac{3}{4})$$

$\therefore y = C.F. + P.I.$

Problem:-02

$2. Solve: (D^3 - D^2 - 6D)y = x^2 + 1.$

$so\ln:$

$A.E., m^3 - 3m^2 - 6m = 0$

$\therefore m = 0, -2, 3.$

$\therefore C.F. = Ae^{0x} + Be^{-2x} + Ce^{3x}$

$$P.I. = \frac{1}{(D^3 - 3D^2 - 6D)}(x^2+1)$$

$$= \frac{1}{-6D[1-(\frac{D^3-D^2}{6D})]}(x^2+1)$$

$$= \frac{1}{-6D}[1-(\frac{D^2-D}{6})]^{-1}(x^2+1)$$

$$= \frac{1}{-6D}[1+(\frac{D^2-D}{6})+(\frac{D^2-D}{6})^2+...](x^2+1)$$

$$= -\frac{1}{6D}[x^2+1+\frac{1}{3}-\frac{x}{3}+\frac{1}{18}]$$

$$= -\frac{1}{6D}[x^2+\frac{25}{18}-\frac{x}{3}]$$

$$= -\frac{1}{6}[\frac{x^3}{3}+\frac{25}{18}x-\frac{x^2}{6}]$$

$\therefore y = C.F. + P.I.$

EXERCISE

$1. Solve: (D^2+4)y = x^4 + \cos^2 x$

$2. Solve: (D^2 - 4D + 3)y = \cos 2x + 2x^2$

Type-IV

$If\, f(x) = e^{ax} x,$ where x is $\sin ax (or) \cos ax (or) x^n$ then $P.I. = \frac{1}{\phi(D)} e^{ax} X = e^{ax} \frac{1}{\phi(D+a)} X$

Here $\frac{1}{\phi(D+a)} X$ can be evaluate by using any one of the first three types.

Problem:-01

Find the P.I.of $(D^2 - 4D) y = xe^x$

$Soln:$

$$P.I. = \frac{1}{D^2 - 4D} xe^x$$

$$= e^x \frac{1}{(D+1)^2 - 4(D+1)} x (\because Re\, place D \to D+1)$$

$$= e^x \frac{1}{D^2 - 2D - 3} x$$

$$= e^x \frac{1}{-3[1 - (\frac{D^2 - 2D}{3})]} x$$

$$= \frac{e^x}{-3} [1 - (\frac{D^2 - 2D}{3})]^{-1} (x)$$

$$= \frac{e^x}{-3} [1 + (\frac{D^2 - 2D}{3})] \, (x)$$

$$= \frac{e^x}{-3} [x + (-\frac{2}{3})]$$

$$= -\frac{e^x}{3} (x - \frac{2}{3})$$

Problem:02

$(D^2+4D+3)y = e^{-x}\sin x$

$So\ln:$

$A.E. m^2+4m+3=0$

$m=-1,-3$

$C.F. = Ae^{-x}+Be^{-3x}$

$$P.I. = \frac{1}{D^2+4D+3}e^{-x\sin x}$$

$$= e^{-x}\frac{1}{(D-1)^2+4(D-1)+3}\sin x (\because \text{Re } place D \to D-1)$$

$$= e^{-x}\frac{1}{D^2+2D}\sin x$$

$$= e^{-x}\frac{1}{2D-1}\sin x$$

$$= e^{-x}\frac{2D+1}{4D^2-1}(\sin x)$$

$$= \frac{e^{x}}{-5}[2\cos x+\sin x]$$

$\therefore y = C.F. + P.I.$

EXERCISE

$Solve: (D^2-4D+13)y = e^{2x}\cos 3x + (x^2+x+9)$

Type – V

To find P.I. when f(x)= $x^n \sin ax (or) x^n \cos ax.$

$$\frac{1}{f(D)} x^n \sin ax (or) x^n \cos ax$$

$$now, \frac{1}{f(D)} x^n (\cos ax + i \sin ax)$$

P.I.= $= \frac{1}{f(D)} x^n e^{iax} = e^{iax} \frac{1}{f(D+ia)} x^n$

$$\therefore \frac{1}{f(D)} x^n \sin ax = I.P. of e^{iax} \frac{1}{f(D+ia)} x^n$$

$$\frac{1}{f(D)} x^n \cos ax = R.P. of e^{iax} \frac{1}{f(D+ia)} x^n$$

Problem:-01

Solve: $(D^2-2D+1)y = x\sin x.$

$$So\ln:$$

$$A.E. m^2-2m+1=0$$

$$m=1,1$$

$$C.F.=e^x(Ax+B)$$

$$P.I.=\frac{1}{D^2-2D+1}x\sin x$$

$$=I.P.of e^{ix}\frac{1}{(D+i)^2-2(D+i)+1}x$$

$$=I.P.of e^{ix}\frac{1}{D^2-2(1-i)D-2i}x$$

$$=I.P.of e^{ix}\frac{1}{-2i[1-(\frac{D^2-2(1-i)D}{2i})]}x$$

$$=I.P.of e^{ix}\frac{i}{2}[1-(\frac{D^2-2(1-i)D}{2i})]^{-1}x$$

$$=I.P.of e^{ix}\frac{i}{2}[1+(\frac{D^2-2(1-i)D}{2i})]\ x$$

$$=I.P.of e^{ix}\frac{i}{2}[x+i+1]$$

$$=I.P.of(\cos x+\sin x)\frac{i}{2}[x+i+1]$$

$$=\frac{1}{2}(x\cos x)+\frac{1}{2}(\cos x)-\frac{\sin x}{2}$$

$$\therefore y=C.F.+P.I.$$

EXERCISE

$(D^2-1)y = x\sin 3x+\cos x.$

LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

[CAUCHY'S HOMOENEOUS EQUATION]

Any equation of the form,

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} \ldots + a_n y = f(x), \rightarrow (1)$$

where $a_1, a_2, \ldots, a_n$ are constants and f(x) is a function of x is called a linear DE with variable coefficients. Equation(1) can be reduce to a linear DE with constant coefficients by putting the substitution.

$$x = e^z (or) z = \log x$$

$$put,$$

$$x\frac{dy}{dx} = D'y \rightarrow (2), x^2 \frac{d^2 y}{dx^2} = D'(D'-1)y \rightarrow (3)$$

$$x^3 \frac{d^3 y}{dx^3} = D'(D'-1)(D'-2)y \rightarrow (4)$$

Sub equation 2,3,4 ...in equation (1), we get a DE with constant coefficients and can be solved by any one of the known methods.

Problem:-01

Reduce the equation $(x^2 D^2 + xD + 1)y = \log x$ in to an ordinary differential equation with constant co- efficient.

Solution:

$$(x^2D^2 + xD + 1)y = \log x$$
$$take, x = e^z (or) \log x = z$$
$$xD = D'; x^2D^2 = D'(D'-1) where, D' = \frac{d}{dz}$$
$$(D'(D'-1) + D'+1)y = z$$
$$(D^2 + 1)y = z$$

which is the required ordinary differential equation with constant co efficient.

Problem:-02

Transform the equation $x^2y'' + xy' = x$ into a linear differential equation with constant co-efficient.

Solution:-

$$(x^2D^2 + xD)y = x$$
$$take, x = e^z (or) \log x = z$$
$$xD = D'; x^2D^2 = D'(D'-1) where, D' = \frac{d}{dz}$$
$$(D'(D'-1) + D')y = e^z$$
$$(D^2)y = e^z$$

Which is the required linear Which is the required differential equation with constant co-efficient.

Problem:-03

Solve: $(x^2D^2 + 4xD + 2)y = 0$

$So\ln:$

$put, x = e^z, z = \log x$

$xD = D', x^2D^2 = D'(D'-1), D' = \frac{d}{dz}$

$[D'(D'-1) + 4D' + 2]y = 0$

$(D'^2 + 3D' + 2)y = 0$

$A.E. m^2 + 3m + 2 = 0$

$m = -1, -2$

$y = Ae^{-z} + Be^{-z}, z = \log x.$

Problem:-04

Solve: $(x^2D^2 - xD - 2)y = x^2 \log x$

$So\ln:$

$put, x = e^z, z = \log x$

$xD = D', x^2D^2 = D'(D'-1), D' = \frac{d}{dz}$

$[D'(D'-1) - D'-2]y = e^{2z}z$

$(D'^2 - 2D'-2)y = e^{2z}z$

$A.E. m^2 - 2m - 2 = 0$

$m = 1 \pm \sqrt{3}$

$C.F. = Ae^{(1+\sqrt{3})z} + Be^{(1-\sqrt{3})z}$

$$P.I. = \frac{1}{D'^2 - 2D'-2} ze^{2z}$$

$$= e^{2z} \frac{1}{(D'+2)^2 - 2(D'+2) - 2} z$$

$$= e^{2z} \frac{1}{D'^2 + 2D'-2} z$$

$$= e^{2z} \frac{1}{-2[1 - (\frac{D'^2 + 2D'}{2})]} z$$

$$= -\frac{e^{2z}}{2}[1 - (\frac{D'^2 + 2D'}{2})]^{-1} z$$

$$= -\frac{e^{2z}}{2}[1 + (\frac{D'^2 + 2D'}{2})]\ z$$

$$= -\frac{e^{2z}}{2}[z + 1]$$

$$= -\frac{e^{2z}}{2}[\log x + 1]$$

$\therefore y = y_C + y_P.$

Problem:05

Solve: $(x^2D^2 - 3xD + 5)y = x^2 \sin(\log x)$

$Soln:$

$put, x = e^z, z = \log x$

$xD = D', x^2D^2 = D'(D'-1), D' = \frac{d}{dz}$

$[D'(D'-1) - 3D' + 5]y = e^{2z} \sin z$

$(D'^2 - 4D' = 5)y = e^{2z} \sin z$

$A.E. m^2 - 4m + 5 = 0$

$m = 2 \pm i$

$C.F. = e^{2z}[A\cos z + Bsinz]$

$P.I. = \frac{1}{D'^2 - 4D' + 5} e^{2z} \sin z$

$= e^{2z} \frac{1}{(D'+2)^2 - 4(D'+2) + 5} \sin z$

$= e^{2z} \frac{1}{D'^2 + 1} \sin z$

$= e^{2z} \frac{1}{-1+1} \sin z$

$= e^{2z} \frac{z}{2D'} \sin z$

$= \frac{ze^{2z}}{2}(-\cos z) where z = \log x$

$\therefore y = y_C + y_P.$

EXERCISE

1. Solve: $(x^2D^2 - xD + 1)y = (\frac{\log x}{x})^2.$

2. Solve: $(x^2D^2 - 3xD + 4)y = x^2 \cos x(\log x).$

Exercise

Solve the following DE's

1. $(e^y+1)\cos x\, dx + e^y \sin x\, dy = 0$ Ans. $\sin x\,(e^y+1) = c$
2. $y' = e^x e^y$ Ans. $e^{-y} + e^x + c = 0$
3. $y' = 1 + x + y + xy$ Ans. $\log(1+y) = x + x^2/2 + c$

EXERCISE

Solve the following DE's

1. $\frac{dy}{dx} - \sin 2x = y \cot x.$

2. $(1 + x^3)\frac{dy}{dx} + 3x^2 y = \sin^2 x.$

3. $\frac{dy}{dx} + y = x$ Ans. $y = x - 1 + ce^{-x}$

4. $\frac{dx}{dy} - \frac{x}{y} = 2y^2$ Ans. $\frac{x}{y} = y^2 + c$

Series Solution and Special Functions

INTRODUCTION

Generally the solutions of ordinary differential equations are obtainable in explicit form called a closed form of the solution. However, many differential equations arising in physical problems are linear but have variable coefficients and do not permit a general solution in terms of known functions. For such equations, it is easier to find a solution in the form of an infinite convergent series called power series solution. The series solution of certain differential equations give rise to special functions such as Bessel's functions, Legendre's polynomials, Lagurre's polynomial, Hermite's polynomial, Chebyshev polynomials. Strum-Liovelle problem based on orthogonality of functions is also included which shows that Bessel's, Legendre's and other equations can be determined from a common point of view.

POWER SERIES SOLUTION OF DIFFERENTIAL EQUATIONS

Consider the differential equation

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\,y = 0 \qquad \ldots (1)$$

where $P_i's$ are polynomials in x.

If $P_0(a) \neq 0$, then $x = a$ is called an ordinary point of (1), otherwise a singular point. Ordinary point is also called a regular point of the equation.

A singular point $x = a$ of (1) is called regular singular point if, (1) can be put in the form

$$\frac{d^2y}{dx^2} + \frac{Q_1(x)}{(x-a)}\frac{dy}{dx} + \frac{Q_2(x)}{(x-a)^2}y = 0 \qquad \ldots (2)$$

provided $Q_1(x)$ and $Q_2(x)$ both possess derivatives of all orders in the neighborhood of a.

A singular point which is not regular is called an irregular singular point.

Note: The power series method sometimes fails to yield a solution

e.g. $x^2y'' + xy' + y = 0$ …(3)

dividing by x^2throughout, $x^2y'' + xy' + y = 0$ …(4)

Here neither of the terms $P_1(x) = \frac{1}{x}$ and $P_2(x) = \frac{1}{x^2}$ is defined at $x = 0$, so we cannot find a power series representation for $P_1(x)$ or $P_2(x)$ that converges in an open interval containing $x = 0$.

Theorem I: If $x = a$ is an ordinary point of the differential equation (1), i.e. $P_0(a) \neq 0$, then series solution of (1) can be found as:

$$y = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots \qquad \ldots (5)$$

Calculate the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ from (5), and substitute the values of y and its derivatives in differential equation (1).

The values of the constants $a_2, a_3, a_4, \cdots$ are obtained by equating to zero the coefficients of various powers of x.

Putting the values of these constants in the solution (5), the desired power series solution of (1) is obtained with a_0, a_1 as its arbitrary constants.

Theorem II: When $x = a$ is a regular singularity of (1) at least one of the solutions can be expressed as,

$$y = (x-a)^m [a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots] \qquad \ldots(6)$$

Theorem III:

The series (5) and (6) are convergent at every point within the circle of convergence at a. A solution in series will be valid only if the series is convergent.

Example 1: Solve in series the equation $\frac{d^2y}{dx^2} - xy = 0$.

Solution: Given differential equation is

$$\frac{d^2y}{dx^2} - xy = 0 \qquad \ldots(1)$$

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\, y = 0$$

Here $P_0(x) = 1$, so $P_0(0) = 1$, i.e. $x = 0$ is the ordinary point of the differential equation (1).

Let the solution of differential equation (1) be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots \qquad \ldots(2)$$

To find a_n

Differentiating (2) w.r.t. x,

$$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots \qquad \ldots(3)$$

Again differentiating w.r.t x

$$\frac{d^2y}{dx^2} = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots \qquad \ldots(4)$$

Substitute values of y from (2) and its derivative from (4) in the differential equation (1), we get

$$(2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + \cdots)$$
$$-x(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots) = 0$$

$$\Rightarrow 2a_2 + (6a_3 - a_0)x + (12a_4 - a_1)x^2 + (20a_5 - a_2)x^3 + \cdots = 0$$

$$2a_2 + (6a_3 - a_0)x + (12a_4 - a_1)x^2 + (20a_5 - a_2)x^3 + \cdots = 0 + 0x + 0x^2 + 0x^3 + 0x^4 + 0x^5 + \ldots\ldots$$

Equating each of the coefficients to zero, we obtain the identities,

$$2a_2 = 0,\quad 6a_3 - a_0 = 0,\quad 12a_4 - a_1 = 0, 20a_5 - a_2 = 0$$

which further gives $\quad a_2 = 0, \quad a_3 = \frac{1}{6}a_0, \quad a_4 = \frac{1}{12}a_1, \quad a_5 = \frac{1}{20}a_2 = 0$

Generalizing the results, $a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}$ … (5)

Putting $n = 4, 5, 6 \ldots$ in (5), we get

$$a_6 = \frac{1}{(6)(5)} a_3 = \frac{1}{(6)(6)(5)} a_0 = \frac{1}{180} a_0,$$

$$a_7 = \frac{1}{(7)(6)} a_4 = \frac{1}{(12)(7)(6)} a_1 = \frac{1}{504} a_1,$$

$$a_8 = 0.$$

Using the values of the constants in (2), the general solution of differential equation (1) becomes

$$y = a_0\left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots\right) + a_1\left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots\right).$$

Example 2:

ASSIGNMENT

Solve the following differential equations in series

1. $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$
2. $\frac{d^2y}{dx^2} + xy = 0.$
3. $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0.$
4. $\frac{d^2y}{dx^2} + y = 0,$ given $y(0) = 0.$
5. $(1 - x^2)y'' + 2y = 0, given\ y(0) = 4,\ y'(0) = 5$

ANSWERS

1. $y = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{2.4} - \frac{x^6}{2.4.6} + \cdots\right) + a_1\left(x - \frac{x^3}{3} + \frac{x^5}{3.5} - \frac{x^7}{3.5.7} + \cdots\right)$
2. $y = a_0\left(1 - \frac{1}{3!}x^3 + \frac{1.4}{6!}x^6 - \frac{1\cdot4.7}{9!}x^9 \ldots\right)$
   $+a_1\left(x - \frac{1.2}{4!}x^4 + \frac{2.7}{7!}x^7 + \cdots\right)$
3. $y = a_0(1 - 2x^2) + a_1x\left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{3}{6}\cdot\frac{x^6}{8} - \frac{5\cdot3}{8\cdot6}\cdot\frac{x^8}{8} - \cdots\right)$
4. $y = a_0\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$
5. $y = 4 + 5x - 4x^2 - \frac{5}{3}x^3 - \frac{x^5}{3} - \frac{x^7}{7} - \cdots$

FROBENIUS METHOD

This method is named after a German mathematician F.G. Frobenius (1849 – 1917) who is known for his contributions to the theory of matrices and groups. This method is employed to find the power series solution of the differential equation

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\,y = 0 \qquad \dots (1)$$

when $x = 0$ is the regular singularity.

Working Procedure

(i) Let $y = x^m(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots)$ … (2)
be the solution of the differential equation (1), where *m* is some real or complex number.

(ii) Substitute in (1) the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ obtained by differentiating (2).

(iii) Find the indicial equation (*a quadratic equation*) by equating to zero the coefficient of the lowest degree term in *x*.

(iv) Find the values of $a_1, a_2, a_3, \cdots$ in terms of a_0 by equating to zero the coefficients of other powers of *x*.

(v) Find the roots m_1, m_2(*say*) of the indicial equation. The complete solution depends on the nature of roots of the indicial equation.

Case I: Roots m_1, m_2 are distinct and do not differ by an integer

In this case, the differential equation (1) has two linearly independent solutions of the following forms:

$$y_1 = x^{m_1}(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)$$

$$y_2 = x^{m_1}(b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots)$$

The complete solution of the differential equation is given by

$y = c_1y_1 + c_2y_2$.

Example 3: Solve $4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Solution: Given $4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ … (1)

$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)\,y = 0$

$P_0(x)=4x$, $P_0(0)=0$,

Here $x = 0$ is a singular point,

Let its solution be

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + a_4 x^{m+4} + \ ... \qquad ...(2)$$

To find a_n

From equation (2)

$$\frac{dy}{dx} = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1}$$

$$+(m+3) a_3 x^{m+2} + \ ... ... \qquad ...(3)$$

$$\frac{d^2y}{dx^2} = m(m-1) a_0 x^{m-2} + (m+1)m\, a_1 x^{m-1}$$

$$+(m+2)(m+1) a_2 x^m + \ ... ... \qquad ...(4)$$

Putting the above values in equation (1), we get

$$4x[m(m-1) a_0 x^{m-2} + (m+1)m\, a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + \ ... ...]$$

$$+2[m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \ ... ...]$$

$$+[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + a_4 x^{m+4} + \ ... ...] = 0 ......(5)$$

Equating the coefficients of x^{m-1} equal to zero

$4m(m-1)a_0 + 2m a_0 = 0$

$\Rightarrow a_0(4m^2 - 4m + 2m) = 0$

Because $a_0 \neq 0$

$\Rightarrow \qquad 4m^2 - 2m = 0$

i.e. $m = 0, \frac{1}{2}$

$\therefore$ The solution of the indicial equation is $m_1 = 0$ and $m_2 = \frac{1}{2}$.

Here, the roots are real, distinct and do not differ by an integer.

$\therefore$ Its solution is $y = c_1 y_1 + c_2 y_2 \qquad ...(6)$

On equating coefficients of x^m, we get

$4(m+1)m a_1 + 2(m+1)a_1 +$ or

$2(m+1)(2m+1)a_1 = -a_0$

$$\Rightarrow \qquad a_1 = \frac{-a_0}{2(m+1)(2m+1)} \qquad ...(7)$$

On equating coefficients of x^{m+1}, we get

Likewise, $4(m+2)(m+1)a_2 + 2(m+2)a_2 + a_1 = 0$

$(m+2)(4m+4+2)a_2 = -a_1$or

$2(m+2)(2m+3)a_2 = -a_1$

$$\Rightarrow a_2 = \frac{-a_1}{2(m+2)(2m+3)} = \frac{a_0}{2^2(m+2)(m+1)(2m+1)(2m+3)} \qquad ...(8)$$

On equating coefficients of x^{m+2}, we get

$)a_3 + a_2 = 0$

$4(m+3)(m+2)a_3 + 2(m+3$

$(m+3)(4m+8+2)a_3 = -a_2$

$$2(m+3)(2m+5)a_3 = \frac{-a_0}{2^2(m+2)(m+1)(2m+1)(2m+3)}$$

$$\Rightarrow \qquad a_3 = \frac{-a_0}{2^3(m+3)(m+2)(m+1)(2m+1)(2m+3)(2m+5)} \text{ and so on.} \qquad \ldots (9)$$

Thus, for $m = 0$, in (2), we get

$$y_{(m=0)} = y_1 = [x^m(a_0 + a_1x + a_2x^2 + \ldots\ldots)]_{m=0}$$

$$= a_0\left[1 - \frac{1}{2}\frac{x}{1.1} + \frac{1}{2^2}\frac{x^2}{2.1.1.3} - \frac{1}{2^3}\frac{x^3}{3.2.1.1.3.5} + \ldots\ldots\right]$$

$$= a_0\left[1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \ldots\ldots\right] = a_0 \cos\sqrt{x} \qquad \ldots (10)$$

Likewise for $m = \frac{1}{2}$, in (2), we get

$$y_{(m=\frac{1}{2})} = y_2 = a_0x^{\frac{1}{2}}\left[1 - \frac{1}{2^1}\frac{x}{\frac{3}{2}.2} + \frac{1}{2^2}\frac{x^2}{\frac{5}{2}\frac{3}{2}.2.4} - \frac{1}{2^3}\frac{x^3}{\frac{7}{2}\frac{5}{2}\frac{3}{2}.2.4.6} + \ldots\ldots\right]$$

$$= a_0\left[\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} - \frac{(\sqrt{x})^7}{7!} + \ldots\ldots\right] = a_0 \sin\sqrt{x} \qquad \ldots (11)$$

Hence, on substituting the values of y_1 and $y_2$in equation (3), we get solution as:

$$y = c_1y_1 + c_2y_2 = \left(C_1\cos\sqrt{x} + C_2\sin\sqrt{x}\right).$$

Example 4: Find the series solution of the equation $\boldsymbol{2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1 - x^2)y = 0}$

OR

Solve the equation $\boldsymbol{2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1 - x^2)y = 0}$ **in power series.**

Solution: Given $2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (1 - x^2)y = 0 \qquad \ldots (1)$

Let its solution be

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \ldots\ldots \qquad \ldots (2)$$

So that

$$\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + (m+3)a_3x^{m+2} + \ldots\ldots \qquad \ldots (3)$$

And

$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)(m)a_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots\ldots \qquad \ldots (4)$$

On substituting the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ in the given equation, we get

$$2x^2[m(m-1)a_0x^{m-2} + (m+1)(m)a_1x^{m-1} + (m+2)(m+1)a_2x^m + \ldots\ldots]$$
$$-x[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + (m+3)a_3x^{m+2} +$$
$$+(1-x^2)[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \ldots\ldots] = 0$$

i.e. $[2m(m-1)a_0x^m + 2(m+1)ma_1x^{m+1} + 2(m+2)(m+1)a_2x^{m+1} + \ldots\ldots]$

$-[ma_0x^m + (m+1)a_1x^{m+1} + (m+2)a_2x^{m+2} + \ldots\ldots]$

$+[(a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots\ldots) - (a_0x^{m+2} + a_1x^{m+3} + \ldots\ldots)]$ =0 …(5)

On equating the coefficients of lowest power of x (i.e. x^m) equal to zero on both sides,

$$2m(m-1)a_0 - ma_0 + a_0 = 0 \quad \ldots (6)$$

$\Rightarrow$ $a_0(2m-1)(m-1) = 0$

$\Rightarrow$ Either $a_0 \neq 0$ or $m = 1, \frac{1}{2}.$

y=$c_1y_1+c_2y_2$

Now equating the coefficients of x^{m+1} equal to zero,

Which $2(m+1)m\,a_1 - (m+1)a_1 + a_1 = 0$ implies either $a_1 = 0$ or $m = 0,$ but

$\Rightarrow$ $a_1m(2m-1) = 0$

$m \neq 0,$

$\therefore$ $a_1 = 0$ … (7)

On comparing the coefficients of x^{m+2},

$\Rightarrow$ $2(m+2)(m+1)a_2 - (m+2)a_2 + (a_2 - a_0) = 0$

$\Rightarrow$ $[(2m^2 + 6m + 4) - (m+1)]a_2 = a_0$

$\Rightarrow$ $a_2 = \frac{a_0}{(m+1)(2m+3)}.$ … (8)

Likewise, on comparing the coefficients of x^{m+3},

$2(m+3)(m+2)a_3 - (m+3)a_3 + a_3 - a_1 = 0$

$\Rightarrow$ $[2(m+3)(m+2)a_3 - (m+3) + 1]a_3 = a_1$

$\Rightarrow$ $a_3 = 0$ (since $a_1 = 0$) … (9)

Further, coefficients of x^{m+4},

$2(m+4)(m+3)a_4 - (m+4)a_4 + a_4 - a_2 = 0$

$\Rightarrow$ $[2(m+4)(m+3) - (m+4) + a_2] = a_2$

$(2m^2 + 13m + 21)a_4 = a_2$

$\Rightarrow$ $a_4 = \frac{a_2}{(m+3)(2m+7)}$ and so on … … (10)

Now for $m = 1,$ $a_2 = \frac{a_0}{(1+1)(2.1+3)} = \frac{a_0}{2.5}$ from (8) … (11)

$a_4 = \frac{a_2}{4.9} = \frac{a_0}{2.5.4.9}$ from (10) … (12)

…… …… …… …… ……

For $m = \frac{1}{2}$, $a_2 = \frac{a_0}{\left(\frac{1}{2}+1\right)\left(2.\frac{1}{2}+3\right)} = \frac{a_0}{\left(\frac{3}{2}\right).(4)} = \frac{a_0}{2.3}$ … (13)

$a_4 = \frac{a_2}{(m+1)(2m+7)} = \frac{a_0}{2.3}\frac{1}{\left(\frac{1}{2}+3\right)\left(\frac{2}{2}+7\right)} = \frac{a_0}{2.3.4.7}$ … (14)

…… …… …… …… ……

Thus $y_1 = (y)_{m=1} = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + a_3x^{m+3} + \;\dots\dots$

$$= a_0x\left[1 + \frac{x^2}{2.5} + \frac{x^4}{2.4.5.9} + \frac{x^6}{2.4.5.9.6.13} + \;\dots\dots\right]$$

$$y_2 = (y)_{m=\frac{1}{2}} = a_0x^{\frac{1}{2}}\left[1 + \frac{x^2}{2.3} + \frac{x^4}{2.3.4.7} + \frac{x^6}{2.3.4.5.7.11} + \;\dots\dots\right]$$

Hence $y = C_1y_1 + C_2y_2$.

Case II: Roots m_1, m_2 are equal, *i.e.* $m_1 = m_2$.

In this case, one of the linearly independent solutions y_1 is obtained by substituting $m = m_1$ and the second solution is obtained as

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=m_1}.$$

Thus the complete solution is given by

$$y = c_1y_1 + c_2\left(\frac{\partial y}{\partial m}\right)_{m=m_1}.$$

Example 5: Solve $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$.

Solution: Given $x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$ … (1)

Let its solution be

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \;\dots\dots \qquad \dots (2)$$

$\therefore$ $\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \;\dots\dots$ … (3)

$\Rightarrow$

$$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)(m)a_1x^{m-1} + (m+2)(m+1)x^m + \;\dots\dots \qquad \dots (4)$$

Putting the above values in equation (1), we have

$$x[m(m-1)a_0x^{m-2} + (m+1)(m)a_1x^{m-1} + (m+2)(m+1)a_2x^m + \;\dots\dots]$$

$$+[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \;\dots\dots]$$

$$-[a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \;\dots\dots] \qquad \dots (5)$$

Equating the coefficients of x^{m-1} to zero,

$$[m\,a_0 + m(m-1)a_0] = 0$$

$\Rightarrow$ Either $a_0 = 0$ or $m^2 = 0$

But $a_0 \neq 0 \therefore m = 0,\ 0.$

$y = c_1 y_1 + c_2 \left(\frac{\partial y}{\partial m}\right)_{m=m_1}.$

Now equate the coefficients of x^m on both sides,

$[(m+1)a_1 + m(m+1)a_1 + a_0] = 0$

$(m+1)^2 a_1 + a_0 = 0$

$\Rightarrow \qquad a_1 = -\frac{a_0}{(m+1)^2}.$ … (6)

Next equate the coefficients of x^{m+1} on both sides,

$[(m+2)(m+1)a_2 + (m+2)a_2 - a_1] = 0$

$\Rightarrow \qquad [(m+2)a_2\{m+1+1\} - a_1] = 0$ or $[(m+2)^2 a_2 - a_1] = 0$

$\Rightarrow \qquad a_2 = \frac{a_1}{(m+2)^2} = \frac{a_0}{(m+1)^2(m+2)^2}$ and so on. ... (7)

Putting the values of $a_1, a_2, \ldots$.in the assumed series solution (2),

$y = a_0 x^m \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2(m+2)^2} + \frac{x^3}{(m+1)^2(m+2)^2(m+3)^2} + \ldots\ldots\right]$ … (8)

Differentiating (8) partially with respect to m

$$\frac{\partial y}{\partial m} = a_0 x^m \log x \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2(m+2)^2} + \ldots\ldots\right]$$

$$+ a_0 x^m \left[0 - \frac{2x}{(m+1)^3} - \frac{2x^2}{(m+1)^2(m+2)^2}\left(\frac{2m+3}{(m+1)(m+2)}\right) + \ldots\ldots\right]$$

$$= a_0 x^m \log x \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2(m+2)^2} + \frac{x^3}{(m+1)^2(m+2)^2(m+3)^2} + \ldots\ldots\right]$$

$$-2a_0 x^m \left[\frac{x}{(m+1)^2(m+1)} + \frac{x^2}{(m+1)^2(m+2)^2}\left(\frac{1}{(m+1)} + \frac{1}{m+2}\right) + \right.$$

x3m+12m+22m+32+ ...... … (9)

Now

$y_1 = y_{(m=0)} = a_0 x \left[1 + \frac{x}{1^2} + \frac{x^2}{1^2.2^2} + \ldots\ldots\right]$ … (10)

$y_2 = \left(\frac{\partial y}{\partial m}\right)_{(m=0)} = y_1 \log x - 2a_0 \left[\frac{x}{1!} + \frac{1}{2!^2}\left(1 + \frac{1}{2}\right)x^2 + \frac{1}{3!^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \ldots\ldots\right]$ … (11)

Therefore, the complete solution is

$$y = (C_1 + C_2 \log x)\left[1 + \frac{x}{1!^2} + \frac{x^2}{2!^2} + \frac{x^3}{2!^3} + \ldots\ldots\right]$$

$$-2C_2\left[x+\frac{1}{2!^2}\left(1+\frac{1}{2}\right)x^2+\frac{1}{3!^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^3+\ \ldots\ldots\right].$$

Case III: Roots m_1, m_2 are distinct and differ by an integer.

In this case, assume that $m_1 < m_2$. If some of the coefficient of y series becomes infinite when $m = m_1$, we modify the form of y replacing a_0 by $b_0(m - m_1)$. Then the complete solution is given by

$$y = c_1(y)_{m_2} + c_1\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Example 5: Solve the equation $\boldsymbol{x(1-x)\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0}$

Solution: Given $x(1-x)\frac{dy}{dx} - 3\frac{dy}{dx} + 2y = 0$ … (1)

Let its solution be $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} +\ \ldots\ldots$ … (2)

$\therefore$ $\frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} +\ \ldots\ldots$ … (3)

and $\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-2} + (m+1)(m)a_1x^{m-1}$

$+(m+2)(m+1)a_2x^m +\ \ldots\ldots$ … (4)

On substituting these values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in the given differential equation,

$(x - x^2)[m(m-1)a_0x^{m-2} + (m+1)(m)a_1x^{m-1} + (m+2)(m+1)a_2x^m +\ \ldots\ldots]$
$-3[ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} +\ \ldots\ldots]$ $+2[a_0x^m + a_1x^{m+1} + a_2x^{m+2} +$
$\ldots\ldots = 0$ … (5)

On equating the coefficients of lowest power of x (i.e.x_{m-1}) on both sides,

$[a_0m(m-1) - 3a_0] = 0$ or $a_0[m(m-4)] = 0$

$\Rightarrow$ Either $a_0 = 0$ or $m(m-4) = 0$

But as $a_0 \neq 0 \therefore m = 0, 4$

$$y = c_1(y)_{m_2} + c_1\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Likewise, equate the coefficients of x^m, x^{m+1}, x^{m+2} equal to zero, and find out the values of unknowns a_0, a_1, a_2 etc.

$\Rightarrow$

For the coefficients of x^m,

$\Rightarrow$ $[-m(m-1)a_0 - 3(m+1)a_1 + (m+1)m$ $a_1 + 2a_0] = 0$ … (6)

$(m-3)(m+1)a_1 = (m-2)(m+1)a_0$

For the coefficient of x^{m+1},

$$[-(m+1)ma_1 + (m+2)(m+1)a_2 - 3(m+2)a_2 + 2a_1] = 0$$

$\Rightarrow$ $$[(m+2)(m-2)]a_2 = (m-1)(m+2)a_1$$

$\Rightarrow$ $$a_2 = \frac{(m-1)}{(m-2)} a_1 = \frac{(m-1)}{(m-3)} a_0 \qquad \dots (7)$$

Similarly,

$$\left.\begin{aligned} a_3 &= \frac{m}{(m-1)} a_2 = \frac{m}{(m-1)}\frac{(m-1)}{(m-3)} a_0 = \frac{m}{(m-3)} a_0 \\ a_4 &= \frac{(m+1)}{m} a_3 = \frac{(m+1)}{m}\frac{m}{(m-3)} a_0 = \frac{(m+1)}{(m-3)} a_0 \\ a_5 &= \frac{(m+2)}{(m+1)} a_4 = \frac{(m+2)}{(m+1)}\frac{(m+1)}{(m-3)} a_0 = \frac{(m+2)}{(m-3)} a_0 \quad \dots \text{so on} \end{aligned}\right\} \qquad \dots (8)$$

$\therefore$ $$y = a_0 x^m \left[1 + \frac{(m-2)}{(m-3)}x + \frac{(m-1)}{(m-3)}x^2 + \frac{m}{(m-3)}x^3 + \frac{(m+1)}{(m-3)}x^4 + \dots\dots\right] \qquad \dots (9)$$

Now, $y_1 = (y)_{m=0} = a_0\left[1 + \frac{2}{3}x + \frac{1}{3}x^2 - \frac{1}{3}x^4 - \dots\dots\right]$ and

$y_2 = (y)_{m=4} = a_0 x^4\left[1 + \frac{2}{1}x + \frac{3}{1}x^2 + \frac{4}{1}x^3 + \frac{5}{1}x^4 + \dots\dots\right]$

Hence the complete solution, $y = c_1 y_1 + c_2 y_2$.

ASSIGNMENT

Use Frobenius method to solve the following differential equations:

1. $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$
2. $4x\frac{d^2y}{dx^2} + 2(1-x)\frac{dy}{dx} - y = 0$
3. $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0$
4. $x(1-x)\frac{d^2y}{dx^2} - (1+3x)\frac{dy}{dx} - y = 0$
5. $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + xy = 0$
6. $2x^2y'' + xy' - (x+1)y = 0$
7. $2x(1-x)\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + 3y = 0$

ANSWERS

1. $y = C_1\left[1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \dots\dots\right]$
   $+C_2 x^{7/3}\left[1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 + \dots\dots\right]$
2. $y = C_1\left[1 + \frac{1}{2.1!}x + \frac{1}{2^2.\ 2!}x^2 + \frac{1}{2^3.\ 3!}x^3 + \dots\dots\right]$
   $+C_2 x^{\frac{1}{2}}\left[1 + \frac{1}{1.3}x + \frac{1}{1.3.5}x^2 + \frac{1}{1.3.5.7}x^3 + \dots\dots\right]$

3. $y = (C_1 + C_2 \log x)\left[1 - \frac{1}{2^2}x^2 + \frac{1}{2^2.4^2}x^4 - \frac{1}{2^2.4^2.6^2}x^6 + \; \dots\dots\right]$
$+C_2\left[\frac{1}{2^2}x^2 - \frac{1}{2^2.4^2}\left(1+\frac{1}{2}\right)x^4 + \frac{1}{2^2.4^2.6^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^6 + \; \dots\dots\right]$

4. $y = (C_1 + C_2 \log x)[1.2x^2 + 2.3x^3 + 3.4x^4 + \; \dots\dots]$
$+C_2[-1 + x + 5x^2 + 11x^3 + \; \dots\dots]$

$y = x^{-1}(a_0 \cos x + a_1 \sin x)$ 5. 6.

$y = a_0 x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \cdots\right) + \frac{a_1}{\sqrt{x}}\left(1 - x - \frac{x^2}{2} + \cdots\right)$ 7.

$y = a_0\sqrt{x}(1 - x) + a_1\left(1 - 3x + \frac{3x^2}{1.3} + \frac{3x^3}{3.5} + \frac{3x^4}{5.7} + \cdots\right)$

BESSEL’S EQUATION

In applied mathematics, many physical problems involving vibrations or heat conduction in cylindrical regions give rise the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \dots (1)$$

which is known as the ***Bessel’s differential equation of order n***. The particular solutions of this differential equation are called ***Bessel’s functions of order n***.

Let $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \; \dots\dots$

$\therefore \quad \frac{dy}{dx} = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \; \dots\dots$

$\frac{d^2y}{dx^2} = m(m-1)a_0x^{m-1} + (m+1)(m)a_1x^m$
$+(m+2)(m+1)a_2x^m + \; \dots\dots$

Putting these in the given differential equation, we get

$\Rightarrow x^2[m(m-1)a_0x^{m-1} + (m+1)(\,m)a_1x^m + (m+2)(m+1)a_2x^m + \; \dots\dots] + x[ma_0x^{m-1} +$
m+1a1xm+m+2a2xm+1+ ......+ [x2−n2a0xm+a1xm+1+a2xm+2+ ......=0

Equating to zero, the coefficient of lowest degree term in x, i.e. x^m

$$m(m-1)a_0 + ma_0 - n^2a_0 = 0, \quad a_0 \neq 0$$

$\therefore$ Indicial equation $\quad [m(m-1) + m] - n^2 = 0$

$\Rightarrow \quad m^2 - n^2 = 0 \therefore m = \pm n$

Now coefficients of x^{m+1}:

$\Rightarrow \quad (m+1)ma_1 + (m+1)a_1 - n^2a_1 = 0$ is fixed

Coefficient $\;[(m+1)^2 - n^2]a_1 = 0\;$ of x^{m+2}:

$(m+2)(m+1)a_2 + (m+2)a_2 - n^2a_2 + a_0 = 0$

$[(m+2)^2 - n^2]a_2 + a_0 = 0$

$\therefore \quad a_2 = -\frac{a_0}{(m+2)^2 - n^2}$

Similarly, $[(m+3)^2 - n^2]a_3 + a_1 = 0$

$$a_3 = -\frac{a_1}{(m+3)^2-n^2} = 0, \qquad \text{as } a_1 = 0$$

So $$a_1 = a_3 = a_5 = \dots\dots = 0$$

$$a_4 = -\frac{a_2}{(m+4)^2-n^2} = \frac{a_0}{[(m+2)^2-n^2][(m+4)^2-n^2]}$$

So $$y = a_0 x^m \left[1 - \frac{x^2}{(m+2)^2-n^2} + \frac{x^4}{[(m+2)^2-n^2][(m+4)^2-n^2]} - \right] \qquad \dots (2)$$

Case 1: For $n = 0,\ m = 0$ as $m = \pm n$

$$y_I = a_0\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \dots\dots\right]$$

$$\frac{\partial y}{\partial m} = y\log x + a_0 x^m\left[-\frac{x^2}{(m+2)^2-n^2}\left\{\frac{-2}{(m+2)^2-n^2}\right\} + \frac{x^4}{[(m+2)^2-n^2][(m+4)^2-n^2]}\left\{\frac{-2}{(m+2)^2-n^2} - \frac{-2}{(m+4)^2-n^2}\right\}\right] + \dots\dots$$

$$y_{II} = \left(\frac{\partial y}{\partial m}\right)_{m=0} = y_I \log x + a_0\left[-\frac{x^2}{2}.\frac{-2}{2^2} + \frac{x^4}{2^2.4^2}\left\{\frac{-2}{2^2} - \frac{-2}{4^2}\right\} + \dots\dots\right]$$

So, the solution is $y = C_1 y_I + C_2 y_{II}$

$$y = (C_1 a_0 + C_2 \log x)\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \dots\dots\right]$$
$$+C_2 a_0\left[\frac{x^2}{2}.\frac{2}{2^2} - \frac{2x^4}{2^2.4^2}\left\{\frac{1}{2^2} + \frac{2}{4^2}\right\} + \dots\dots\right]$$

Case 2: For n non integral and equal to $n(n = m)$ replace a_0 in equation (2) by $\frac{1}{2^n\sqrt{n+1}}$

$$y_0 = \frac{1}{2^n\sqrt{n+1}}x^n\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^2(n+1)4.2(n+2)} + \dots\dots\right]$$

We get
$$= \left(\frac{x}{2}\right)^n\left[\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\sqrt{n+3}}\left(\frac{x}{2}\right)^4 + \dots\dots\right]$$

$$= \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\sqrt{n+r+1}}\left(\frac{x}{2}\right)^{n+2r} = J_n(x)$$

i.e. $$J_n(x) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\sqrt{n+r+1}}\left(\frac{x}{2}\right)^{n+2r} \qquad \dots (3)$$

Similarly by putting $m = -n$, we get the other solution

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\sqrt{-n+r+1}}\left(\frac{x}{2}\right)^{-n+2r}$$

The resulting solution is

$$y = C_1 J_n(x) + C_2\ J_{-n}(x)$$

$J_n(x)$&$J_{-n}(x)$ as defined as above.

Case 3: If n is integral

Let $y = u(x)J_n(x)$ $\qquad y' = u'(x)J_n + uJ_n'$

$$y'' = u''J_n + 2u'J_n' + uJ_n''$$

Putting these in $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$

$$\Rightarrow \quad x^2(u''J_n + 2u'J_n' + uJ_n'') + x(u'J_n + uJ_n') + (x^2-n^2)uJ_n = 0$$

$$\Rightarrow \quad u\{x^2J_n'' + xJ_n' + (x^2-n^2)J_n\} + 2u'x^2J_n' + x^2u''J_n + xu'J_n = 0$$

Now $J_n(x)$ is a solution of $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$

$\therefore \qquad x^2J_n'' + xJ_n' + (x^2 - n^2)J_n = 0$

We get,

$$2u'x^2J_n' + x^2u''J_n + xu'J_n = 0$$

$\Rightarrow \qquad 2\frac{J_n'}{J_n} + \frac{u''}{u'} + \frac{1}{x} = 0 \qquad$ (divide by $J_nu'x^2$)

Integrating $\quad 2\log_e J_n + \log_e u' \log_e x = \log c$

$\Rightarrow \qquad uJ'^2_n x = B \qquad$ Where B is constant of integration.

$\Rightarrow \qquad u' = B\frac{1}{xJ_n^2}$

Integrating $\quad u = A + B\int \frac{dx}{x(J_n(x))^2}$

So the solution of y in this case

$$y = u(x)\,J_n(x)$$

$$= AJ_n(x) + BJ_n(x)\int \frac{dx}{x(J_n(x))^2}$$

$$= AJ_n(x) + By_n(x)$$

where $y_n(x) = J_n(x)\int \frac{dx}{x(J_n(x))^2}$ is the Bessel's function of the second kind and $J_n(x)$ is Bessel's function of first kind.

RECURRENCE FORMULAE FOR $J_n(x)$

The following relations are the recurrence formulae for Bessel's functions and are very useful in the solution of Boundary value problems and in establishing various properties of Bessel's functions:

1. $\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)$
2. $\frac{d}{dx}(x^{-n} J_n(x)) = -x^{-n} J_{n+1}(x)$
3. $J_n(x) = \frac{x}{2n}(J_{n-1}(x) + J_{n+1}(x))$
4. $J_n'(x) = \frac{1}{2}(J_{n-1}(x) - J_{n+1}(x))$
5. $J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$
6. $J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$

EXPANSION FOR J_0 AND J_1

We know that $\quad J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\sqrt{n+r+1}}\left(\frac{x}{2}\right)^{n+2r}$

Taking $n = 0$ and 1in above Bessel's function, we get

$$J_0(x) = 1 - \frac{1}{1!}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

and $$J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\;4!}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

VALUE OF $J_1(x)$

$\bar{2}$

In Bessel's functions, the function $J_{1/2}$ is the simplest one, as it can be expressed in finite form. Taking $n = 1/2$ in the value of $J_n(x)$, we get

$$J_{1/2}(x) = \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{1!\ \Gamma\left(\frac{5}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\ \Gamma\left(\frac{7}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= \left(\frac{x}{2}\right)^{1/2}\left[\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} - \frac{1}{\frac{3}{2}\frac{1}{2}\ \Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^2 + \frac{1}{2\frac{5}{2}\frac{3}{2}\frac{1}{2}\ \Gamma\left(\frac{1}{2}\right)}\left(\frac{x}{2}\right)^4 - \cdots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)}\left[\frac{2}{1!} - \frac{2\,x^2}{3!} + \frac{2\,x^4}{5!} - \cdots\right]$$

Now multiplying the series by $\frac{x}{2}$ and outside by $\frac{2}{x}$, we get

$$J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}\,\sqrt{\pi}}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

Similarly, taking $n = 1/2$ in the value of $J_{-n}(x)$, we get

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

GENERATING FUNCTION FOR $J_n(x)$

To prove that $e^{\frac{1}{2}x(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$.

We have $e^{\frac{1}{2}x(t-\frac{1}{t})} = e^{\frac{xt}{2}} \times e^{-\frac{x}{2t}}$

$$= \left[1 + \frac{xt}{2} + \frac{1}{2\,!}\left(\frac{xt}{2}\right)^2 + \frac{1}{3\,!}\left(\frac{xt}{2}\right)^3 + \cdots\right] + \left[1 - \frac{x}{2t} + \frac{1}{2\,!}\left(\frac{x}{2t}\right)^2 - \frac{1}{3\,!}\left(\frac{x}{2t}\right)^3 + \cdots\right]$$

The coefficient of t^nin this product is

$$\frac{1}{n\,!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)\,!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2\,!\,(n+1)\,!}\left(\frac{x}{2}\right)^{n+4} - \cdots = J_n(x)$$

As all the integral powers of *t*, both positive and negative occurs, we have

$$e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = J_0(x) + tJ_1(x) + t^2J_2(x) + t^3J_3(x) + \cdots$$
$$+t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + t^{-3}J_{-3}(x) + \cdots$$
$$= \sum_{n=-\infty}^{\infty} t^n\, J_n(x)$$

Thus the coefficients of different powers of t in the expansion of $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)}$ *give Bessel's functions of various orders. Hence it is known as the generating function of Bessel's functions.*

Example 6: Evaluate $\int_0^{\infty} e^{-ax} J_0(bx)dx$

Solution: We know that

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} cos(n\theta - x\sin\theta)\, d\theta$$

For

$$n = 0, \quad \begin{cases} J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\theta) \quad d\theta \\ or \\ J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta) \\ \quad d\theta \end{cases}$$

$$\Rightarrow \quad J_0(bx) = \frac{1}{\pi}\int_0^{\pi} \cos(bx\sin\theta)d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(bx\sin\theta)d\theta$$

So, $$\int_0^{\infty} e^{-ax} J_0(bx)dx = \int_0^{\infty} e^{-ax}\left[\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(bx\sin\theta)d\theta\right]dx$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\left[\frac{e^{(-a+i\,b\sin\,\theta)x} + e^{(-a-i\,b\sin\,\theta)x}}{2}\right] \text{dx dy}$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left[\frac{e^{(i\,b\sin\,\theta-a)x}}{(i\,b\sin\theta-a)} + \frac{e^{-(i\,b\sin\,\theta+a)x}}{-(i\,b\sin\,\theta+a)}\right]_0^{\infty} d\theta$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left[\frac{-1}{(i\,b\sin\theta-a)} + \frac{1}{(i\,b\sin\,\theta+a)}\right] d\theta$$

$$= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\frac{-2a}{c^2b^2sin^2\theta-a^2}\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{\frac{\pi}{2}}\frac{2a}{a^2+b^2sin^2\theta}\,d\theta$$

$$= \frac{2a}{\pi}\int_0^{\frac{\pi}{2}}\frac{sec^2\theta}{a^2sec^2\theta+b^2tan^2\theta}\,d\theta$$

$$= \frac{2a}{\pi}\int_0^{\frac{\pi}{2}}\frac{sec^2\theta}{(a^2+b^2)tan^2\theta+a^2}\,d\theta$$

Now take $\sqrt{(a^2+b^2)}\tan\theta = t$

$\therefore$ $\sqrt{(a^2+b^2)}\, sec^2\theta\, d\theta = dt$

Further, if $\theta = 0 \quad it\ implies \quad t = 0$

$\theta = \frac{\pi}{2} \quad it\ implies \quad t = \infty$

$\therefore$ $$e^{-ax}J_0(bx)dx = \frac{2a}{\pi}\int_0^{\infty}\frac{1}{t^2+a^2}\frac{dt}{\sqrt{(a^2+b^2)}}$$

$$= \frac{2a}{\pi\sqrt{(a^2+b^2)}} \int_0^\infty \frac{1}{a^2+t^2}\, dt$$

$$= \frac{2a}{\pi\sqrt{(a^2+b^2)}} \left[\frac{1}{a}\tan^{-1}\frac{t}{a}\right]_0^\infty$$

$$= \frac{2}{\pi\sqrt{(a^2+b^2)}} \left[\tan^{-1}\left(\frac{\infty}{a}\right) - \tan^{-1} 0\right]$$

$$= \frac{2}{\pi\sqrt{(a^2+b^2)}} \left(\frac{\pi}{2} - 0\right) = \frac{1}{\sqrt{(a^2+b^2)}}$$

Example 7: Show that $\boldsymbol{J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)}$

Solution: We know

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

∴ for $n = 3$

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x) \qquad (1)$$

For $n = 2$

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x) \qquad (2)$$

For $n = 1$

$$J_2(x) = \frac{2}{x}J_1(x) - J_0(x) \qquad (3)$$

Substituting $J_2(x)$ from (3) in (2), we get

$$J_3(x) = \frac{4}{x}\left\{\frac{2}{x}J_1(x) - J_0(x)\right\} - J_1(x)$$

$$= \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x) \qquad (4)$$

Now substituting for $J_2(x)$ and $J_3(x)$ in (1), we will have

$$J_4(x) = \frac{6}{x}\left\{\left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)\right\} - \left\{\frac{2}{x}J_1(x) - J_0(x)\right\}$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) + \left(1 - \frac{24}{x^2}\right)J_0(x)$$

Example 8: Show that

(i) $\boldsymbol{J_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x)\cot x}$

(ii) $\boldsymbol{J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\sin x + \frac{\cos x}{x}\right)}$

(iii) $\boldsymbol{J_{-\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x}\sin x + \frac{3-x^2}{x^2}\cos x\right]}$

Solution:(i) We know

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x \qquad \text{and} \qquad J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

$$\therefore \quad \frac{J_{-\frac{1}{2}}(x)}{J_{\frac{1}{2}}(x)} = \frac{\sqrt{\frac{2}{\pi x}}\cos x}{\sqrt{\frac{2}{\pi x}}\sin x} = \cot x$$

Hence $J_{-\frac{1}{2}}(x) = J_{\frac{1}{2}}(x)\cot x$

(ii) We know

$$J_{n-1}(x) = \frac{2n}{x}J_n(x) - J_{n+1}(x)$$

$\therefore$ For $n = -\frac{1}{2}$

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}}\cos x - \sqrt{\frac{2}{\pi x}}\sin x$$

$$J_{-\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right]$$

(iii) We know

,

$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$, $\quad J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$

$J_{n-1}(x) = \frac{2n}{x}J_n(x) - J_{n+1}(x)$

$\therefore$ for $n = -\frac{3}{2}$

$$J_{-\frac{5}{2}}(x) = -\frac{3}{x}J_{-\frac{3}{2}}(x) - J_{-\frac{1}{2}}(x) \qquad (1)$$

$$J_{-\frac{3}{2}}(x) = -\frac{1}{x}J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right] \qquad (2)$$

$\therefore$ from (1) and (2), we will have

$$J_{-\frac{5}{2}}(x) = -\frac{3}{x} \times -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x}{x} + \sin x\right] - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3}{x^2} - 1\right)\cos x + \frac{3}{x}\sin x\right]$$

Example 9:Prove that

(i) $\frac{d}{dx}[J_0(x)] = -J_1(x)$,

(ii) $\frac{d}{dx}[x\,J_1(x)] = xJ_0(x)$

(iii) $\frac{d}{dx}[x^n J_n(ax)\,] = a\,x^n J_{n-1}(x)$

(iv) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$

Solutions:(i) We know that

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

For $n = 0$, we will have

$$\frac{d}{dx}[x^0 J_0(x)] = -x^0 J_1(x)$$
$$\frac{d}{dx}[J_0(x)] = -J_1(x)$$

(ii) We know

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

For $n = 1$, it will give

$$\frac{d}{dx}[x\,J_1(x)] = xJ_0(x)$$

(iii) To prove $\frac{d}{dx}[x^n J_n(ax)\,] = a\,x^n J_{n-1}(x)$

Let $ax = t$ or $x = \frac{t}{a}$

$\therefore$ $$x^n J_n(ax) = \left(\frac{t}{a}\right)^n J_n(t)$$

Differentiating with respect to x'', we get

$$\frac{d}{dx}[x^n J_n(ax)] = \frac{d}{dt}\left[\left(\frac{t}{a}\right)^n J_n(t)\right].\frac{dt}{dx}$$
$$= \frac{1}{a^n}.\frac{d}{dt}[t^n J_n(t)].\,a,$$
$$= \frac{1}{a^{n-1}}.\,t^n J_{n-1}(t),$$
$$= \frac{1}{a^{n-1}}.\,(ax)^n J_{n-1}(ax),$$
$$= ax^n J_{n-1}(ax)$$

(iv) To prove

(v)

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

We know

$$J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\sqrt{(n+r+1)}}\left(\frac{x}{2}\right)^{n+2r}$$

$$x^{-n}J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\sqrt{(n+r+1)}}\cdot\frac{1}{2^{n+2r}}\cdot x^{2r}$$

$$\frac{d}{dx}[x^{-n}J_n(x)] = \sum_{r=1}^{\infty}(-1)^r \frac{1}{r!\sqrt{(n+r+1)}}\,\frac{1}{2^{n+2r}}\cdot 2r\,x^{2r-1}$$

$$= -x^{-n}\sum_{r=1}^{\infty}(-1)^{r-1}\frac{1}{(r-1)!\sqrt{\{n+1+(r-1)+1\}}}\cdot\frac{x^{n+1+2r}}{2^{n-1+2r}}$$

Taking $(r-1) = k$

$$= -x^{-n}\sum_{k=0}^{\infty}(-1)^k\cdot\frac{1}{k!\sqrt{(n+1+k+1)}}\cdot\left(\frac{x}{2}\right)^{n+1+2r}$$
$$= -x^{-n}J_{n+1}(x).$$

Example 10: Show by the use of recurrence formula, that

(i) $J_0''(x) = \frac{1}{2}[J_2(x) - J_0(x)]$

(ii) $J_1''(x) = -J_1(x) + \frac{1}{x}J_2(x)$

Solutions: (i) We know

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

∴ for $n = 0$

$$\frac{d}{dx}[J_0(x)] = -J_1(x)$$

Differentiating with respect to x'', we will have

$$\frac{d^2}{dx^2}[J_0(x)] = -\frac{d}{dx}[J_1(x)]$$
$$J_{0''}(x) = -J_1(x)$$

But $$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

∴ for $n = 1$

$$J_1'(x) = \frac{1}{2}[J_0(x) - J_2(x)]$$

∴ $$J_0''(x) = -\frac{1}{2}[J_0(x) - J_2(x)]$$

$$= \frac{1}{2}[J_2(x) - J_0(x)]$$

(ii) We know

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

∴ $$J_1'(x) = \frac{1}{2}[J_0(x) - J_2(x)]$$

Differentiating with respect to x'', we get

$$J_1''(x) = \frac{1}{2}\left[J_0'(x) - J_2'(x)\right]$$

But $J_0'(x) = -J_1(x)$ and also

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$$

For $n = 2$

$$J_2'(x) = J_1(x) - \frac{2}{x}J_2(x)$$

$\therefore$

$$J_1''(x) = \frac{1}{2}\left[-J_1(x) - J_1(x) + \frac{2}{x}J_2(x)\right]$$
$$= \frac{1}{x}J_2(x) - J_1(x)$$

Example 11:Show that

(i) $\boldsymbol{4\,J_0'''(x) + 3\,J_0'(x) + J_3(x) = 0}$

(ii) $\boldsymbol{4\,J_n''(x) = J_{n-2}(x) - 2\,J_n(x) + J_{n+2}(x) = 0}$

Solution:(i)

We know $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

for $n = 0$

$$\frac{d}{dx}[J_0(x)] = -J_1(x)$$

Differentiating with respect to $'x'$, we get

$$J_0''(x) = -J_1'(x)$$

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \quad (1)$$

$\therefore$ for $n = 1$

$$J_1'(x) = \frac{1}{2}[J_0(x) - J_2(x)]$$

Differentiating again, it will give

$$J_0'''(x) = \frac{1}{2}[-J_0'(x) + J_2'(x)]$$

$$= \frac{1}{2}[J_1(x) + J_2'(x)]$$

From (1), for $\boldsymbol{n = 2}$

$$J_2'(x) = \frac{1}{2}[J_1(x) - J_3(x)]$$

$\therefore$ $$J_0'''(x) = \frac{1}{2}\left[J_1(x) + \frac{1}{2}\{J_1(x) - J_3(x)\}\right]$$

$$= \frac{1}{4}[3\,J_1(x) - J_3(x)]$$

$$= \frac{1}{4}[-3\,J_0'(x) - J_3(x)]$$

$\therefore$ $$4\,J_0'''(x) + 3\,J_0'(x) + J_3(x) = 0$$

(ii) We know

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \quad (1)$$

Differentiating with respect to $'x'$, we get

$$J_n''(x) = \frac{1}{2}[J_{n-1}'(x) - J_{n+1}'(x)] \quad (2)$$

From (1) $$J_{n-1}'(x) = \frac{1}{2}[J_{n-2}(x) - J_n(x)]$$

and $$J_{n+1}'(x) = \frac{1}{2}[J_n(x) - J_{n+2}(x)]$$

$\therefore$ From (2), we get

$$J_n''(x) = \frac{1}{2}\left[\frac{1}{2}\{J_{n-2}(x) - J_n(x)\} - \frac{1}{2}\{J_n(x) - J_{n+2}(x)\}\right]$$

$$= \frac{1}{4}\,[J_{n-2}(x) - 2\,J_n(x) + J_{n+2}(x)]$$

$\therefore$ $$4\,J_n''(x) = J_{n-2}(x) - 2\,J_n(x) + J_{n+2}(x)$$

Example 12: Prove that

(i) $\frac{d}{dx}\left[J_n^2(x)\right] = \frac{x}{2n}\left[J_{n-1}^2(x) - J_{n+1}^2(x)\right]$

(ii) $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2\left[\frac{n}{x}J_n^2(x) - \frac{n+1}{2}J_{n+1}^2(x)\right]$

Solutions:(i) LHS $= 2\,J_n(x)\,J_n'(x)$

But $$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

$\Rightarrow$ $$J_n(x) = \frac{x}{2n}[J_{n+1}(x) + J_{n-1}(x)]$$

and $$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

LHS $= 2J_n(x)J_n'(x) = 2.\frac{x}{2n}[J_{n+1}(x) + J_{n-1}(x)] \times \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

$= \frac{x}{2n}[J_{n-1}^2(x) - J_{n+1}^2(x)]$ = RHS

Hence the result

(ii) LHS $= 2J_n(x)J_n'(x) + 2J_{n+1}(x)J_{n+1}'(x)$

But $\quad J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

and $\quad J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$

$\Rightarrow \quad J_{n+1}'(x) = J_n(x) - \frac{n+1}{x} J_{n+1}(x)$

$\therefore$ LHS $= 2\,J_n(x).\left\{\frac{n}{x} J_n(x) - J_{n+1}(x)\right\} + 2\,J_{n+1}(x)\left\{J_n(x) - \frac{n+1}{x} J_{n+1}(x)\right\}$

$= 2\left[\frac{n}{x} J_n^2(x) - J_n(x).J_{n+1}(x) + J_{n+1}(x).J_n(x) - \frac{n+1}{x} J_{n+1}^2(x)\right]$

$= 2\left[\frac{n}{x} J_n^2(x) - \frac{n+1}{x} J_{n+1}^2(x)\right]$ = RHS

Example 13: Prove that

(i) $\int \boldsymbol{J_0(x) J_1(x) = -\frac{1}{2}[J_0(x)]^2}$

(ii) $\int_0^r \boldsymbol{x J_0(ax) = \frac{r}{a} J_1(ar)}$

(iii) $\int_0^\infty \boldsymbol{e^{-ax} J_0(bx) = \frac{1}{\sqrt{a^2+b^2}}}$

Solution:(i) We know $J_0'(x) = -J_1(x)$

$$\int J_0(x) J_1(x) = -\int J_0(x) J_0'(x) dx$$

$\therefore \quad = -\frac{1}{2}\,[J_0(x)]^2$

(ii) Let $ax = t,\ \therefore\ a dx = dt,\quad (0\ to\ r) \rightarrow (0\ to\ ar)$

$\therefore \quad \int_0^r x\,J_0(ax)\,dx = \int_0^{ar} \frac{t}{a} J_0(t).\frac{dt}{a}$

$$= \frac{1}{a^2}\int_0^{ar} tJ_0(t)dt = \frac{1}{a^2}\int_0^{ar} \frac{d}{dt}[t\,J_1(t)]dt$$

$$= \frac{1}{a^2}[t\,J_1(t)]_0^{ar} = \frac{1}{a^2}[ar\,J_1(ar) - 0.x.J_1(0)]$$

$$= \frac{1}{a} r\,J_1(ar)$$

(iii) $\quad \int_0^\infty e^{-ax} J_0(bx)dx$

$$= \int_0^\infty e^{-ax}.\frac{1}{\pi}\int_0^\pi \cos(bx\cos\varphi)d\varphi\;dx$$

Integrating the order of integration, we get

$= \frac{1}{\pi}\int_0^\pi \int_0^\infty e^{-ax}\cos(bx\cos\varphi)\,dx\,d\varphi$

$= \frac{1}{\pi}\int_0^\pi \left[\frac{e^{-ax}}{(a^2+b^2cos^2\varphi)}\{-a\cos(bx\cos\varphi) + b\cos\varphi\sin(bx\cos\varphi)\}\right]_0^\infty d\varphi$

$= \frac{1}{\pi}\int_0^\pi \frac{a}{a^2+b^2cos^2\varphi}d\varphi = \frac{1}{\pi}\int_0^\pi \frac{a\,sec^2\varphi}{a^2sec^2\varphi+b^2}d\varphi = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{a\,sec^2\varphi}{(a^2+b^2)+a^2tan^2\varphi}$

$$= \frac{2}{\pi a}\left[\tan^{-1}\left(\frac{a\tan\varphi}{\sqrt{a^2+b^2}}\right)\right]_0^{\frac{\pi}{2}} \times \frac{a}{\sqrt{a^2+b^2}}$$

$$= \frac{2}{\pi a}.\frac{a}{\sqrt{a^2+b^2}} \times \left(\frac{\pi}{2} - 0\right) = \frac{1}{\sqrt{a^2+b^2}}$$

Example 14: Starting with series with generating functions, prove that

$\boldsymbol{2nJ_n(x) = x[J_{n-1}(x) = x[J_{n-1}(x) + J_{n+1}(x)]]}$ **and**

$\boldsymbol{xJ_n^{'}(x) = n\,J_n(x) - x\,J_{n+1}(x)}$ **(1)**

Solutions: We know $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} t^n J_n(x)$

Differentiating both sides with respect to $'t'$, we get

$$\frac{1}{2}x\left(1+\frac{1}{t^2}\right).\,e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} nt^{n-1}J_n(x)$$

$$\frac{1}{2}x\left(1+\frac{1}{t^2}\right)\sum_{-\infty}^{\infty} t^n J_n(x) = n\sum_{-\infty}^{\infty} t^{n-1}J_n(x)$$

Equating the coefficients of t^{n-1}, we will have

$$\frac{1}{2}x\,J_{n-1}(x) + \frac{1}{2}x\,J_{n+1}(x) = nJ_n(x)$$

$\Rightarrow$ $$2n\,J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)] \qquad (2)$$

Now differentiating with respect to $'x'$, we get

$$\frac{1}{2}\left(t-\frac{1}{t}\right)e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} t^n J_n^{'}(x)$$

$$\frac{1}{2}\left(t-\frac{1}{t}\right)\sum_{-\infty}^{\infty} t^n J_n(x) = \sum_{-\infty}^{\infty} t^n J_n^{'}(x)$$

Equating the coefficients of $'t^n'$, we will have

$$\frac{1}{2}J_{n-1}(x) - \frac{1}{2}J_{n+1}(x) = J_n^{'}(x)$$

$\Rightarrow$ $$J_n^{'}(x) = \frac{1}{2}\left(J_{n-1}(x) - J_{n+1}(x)\right) \qquad (3)$$

From (2), substituting $J_{n-1}(x)$ in (3), we get

$$J_n^{'}(x) = \frac{1}{2}\left[\left\{\frac{2n}{x}J_n(x) - J_{n+1}(x)\right\} - J_{n+1}(x)\right]$$

$\Rightarrow$ $$J_n^{'}(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$$

Example 15: Establish the Jacobi series

$$\boldsymbol{\cos(x\cos\theta) = J_0 - 2J_2\cos 2\theta + 2J_4\cos 4\theta - \dots\dots}$$

$$\boldsymbol{\sin(x\cos\theta) = 2[J_1\cos\theta - J_3\cos 3\theta + J_5\cos 5\theta - \dots\dots]}$$

Solutions: We know $e^{\frac{1}{2}x\left(t-\frac{1}{t}\right)} = \sum_{-\infty}^{\infty} t^n J_n(x)$

$= J_0(x) + \sum_{n=1}^{\infty} J_n(x)\left(t^n + (-1)^n\frac{1}{t^n}\right)$ (1)

Now, let $t = \cos\theta + i\sin\theta$ and $\frac{1}{t} = \cos\theta - i\sin\theta$

To get $t^p = \cos p\theta + i\sin p\theta$ and $t^{-p} = \cos p\theta - i\sin p\theta$

and thus $t^p + t^{-p} = 2\cos p\theta$ and $t^p - t^{-p} = 2\,i\sin p\theta$

$\therefore$ From (1)

$$e^{ix\sin\theta} = J_0(x) + 2i\,J_1(x)\sin\theta + 2J_2(x)\cos 2\theta$$

$$+2iJ_3(x)\sin 3\theta + 2J_4(x)\cos 4\theta + \dots\dots$$

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = \{J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \dots\dots\}$$

$$+i\{2J_1(x)\sin\theta + 2J_3(x)\sin\theta + \dots\dots\}$$

Equating the real and imaginary parts, we get

$$\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta + \dots\dots]$$

and $$\sin(x\sin\theta) = 2\{J_1(x)\sin\theta + J_3(x)\sin 3\theta + \dots\dots\}$$

Replacing θ by $\frac{\pi}{2} - \theta$, we get

$$\cos(x\cos\theta) = J_0(x) - 2\cos 2\theta\ J_2(x) + 2J_4(x)\cos 4\theta + \dots\dots$$

and $$\sin(x\cos\theta) = 2[J_1(x)\sin\theta - J_3(x)\sin 3\theta + \dots\dots]$$

Example 16: Prove that

(i) $\boldsymbol{\sin x = 2[J_1(x) - J_3(x) + J_5(x) - \dots\dots]}$
(ii) $\boldsymbol{\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - 2J_6(x) + \dots\dots}$
(iii) $\boldsymbol{1 = J_0 + 2J_2 + 2J_4 + 2J_6 + \dots\dots}$

Solution:
We know

$$\cos(x\sin\theta) = J_0(x) + 2[J_2(x)\cos 2\theta + J_4(x)\cos 4\theta$$

and $$\sin(x\sin\theta) = 2\{J_1(x)\sin\theta + J_3(x)\sin 3\theta + \dots\dots\}$$

(ii) $$\cos x = J_0(x) + 2\{J_2(x)\cos\pi + J_4(x)\cos\ 2\pi + J_6(x)\cos 3\pi + \dots\dots\}$$
$$= J_0(x) + 2\{-J_2(x) + J_4(x) - J_6(x) + \dots\dots\}$$
$$= J_0(x) - 2J_2(x) + 2J_4(x) - 2J_6(x) + \dots\dots$$

$$\sin x = 2\left\{J_1(x)\sin\frac{\pi}{2} + J_3(x)\sin\frac{3\pi}{2} + J_5(x)\sin\frac{5\pi}{2} + \dots\dots\right\}$$

On taking $\theta = \frac{\pi}{2}$, we will have

(i)

$$= 2\{J_1(x) - J_3(x) + J_5(x) - \dots\dots\}$$

$$\cos 0 = 1 = J_0(x) + 2\{J_2(x)\cos 2\times 0 + J_4(x)\cos 4\times 0 + \dots\dots.\}$$

$\Rightarrow$ $1 = J_0(x) + 2J_2(x) + 2J_4(x) + \dots\dots$ (iii) Taking $\theta = 0$, we get

ASSIGNMENT

1. Compute $J_0(2)$ and $J_1(1)$ correct to three decimal places.
2. Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.
3. Prove that
   (a) $J_n''(x) = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$
   (b) $\frac{d}{dx}[xJ_n(x)J_{n+1}(x)] = x[J_n^2(x) - J_{n+1}^2(x)]$.
4. Prove that $J_{\frac{5}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left\{\frac{3-x^2}{x^2}\sin x - \frac{3}{x}\cos x\right\}$.
5. Prove that
   (a) $\int J_3(x)dx = c - J_2(x) - \frac{2}{x}J_1(x)$
   (b) $\int xJ_n{}^2(x)dx = \frac{1}{2}x^2[J_0{}^2(x) + J_1{}^2(x)]$.
6. Show that
   a) $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)d\theta$, n being an integer.
   b) $J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\cos\theta)d\theta$
   c) $J_0{}^2 + 2J_1{}^2 + 2J_2{}^2 + 2J_3{}^2 + \cdots = 1$.

ANSWERS

1. 0.224, 0.44
2. $J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right)J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right)J_0(x)$

EQUATIONS REDUCIBLE TO BESSEL'S EQUATION

In differential calculus, we come across such differential equations which can be easily reduced to Bessel's equation and thus can be solved by the means of Bessel's functions. The following are some examples of such differential equations:

1. ***Reduce the differential equation*** $\boldsymbol{x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (k^2x^2 - n^2)y = 0}$ ***to the Bessel's Equation.***

Putting $t = kx$, so that $\frac{dy}{dx} = k\frac{dy}{dt}$ and $\frac{d^2y}{dx^2} = k\frac{d^2y}{dt^2}$ in the above differential equation, we get

$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0$, which is the Bessel's Form of Equation.

∴ Its solution is $y = c_1J_n(t) + c_2J_{-n}(t)$, n is non-integral. or
$y = c_1J_n(t) + c_2Y_n(t)$, n is integral.

Hence solution of the given differential equation is

$y = c_1 J_n(kx) + c_2 J_{-n}(kx),\ n$ is non-integral.

or $y = c_1 J_n(kx) + c_2 Y_n(kx),\ n$ is integral.

2. ***Reduce the differential equation*** $\boldsymbol{x\frac{d^2y}{dx^2} + a\frac{dy}{dx} + k^2xy = 0}$ ***to the Bessel's Equation.***

Putting $y = x^n z$, so that $\frac{dy}{dx} = x^n\frac{dz}{dx} + nx^{n-1}z$

and $\frac{d^2y}{dx^2} = x^n\frac{d^2z}{dx^2} + 2nx^{n-1}\frac{dz}{dx} + n(n-1)x^{n-2}z$ in the above differential equation, we get

$x^{n+1}\frac{d^2z}{dx^2} + (2n+a)x^n\frac{dz}{dx} + (k^2x^2 + n^2 + (a-1)n)x^{n-1}z = 0$

Dividing throughout by x^{n-1} and putting $2n + a = 1$, we get

$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + (k^2x^2 - n^2)z = 0$, which is the Bessel's Form of Equation.

And its solution is $z = c_1 J_n(kx) + c_2 J_{-n}(kx),\ n$is non-integral.

or $y = c_1 J_n(kx) + c_2 Y_n(kx),\ n$ is integral.

Hence solution of the given differential equation is

$y = x^n[c_1 J_n(kx) + c_2 J_{-n}(kx)],\ n$ is non-integral.

or $y = x^n[c_1 J_n(kx) + c_2 Y_n(kx)],\ n$ is integral.

3. ***Reduce the differential equation*** $\boldsymbol{x\frac{d^2y}{dx^2} + c\frac{dy}{dx} + k^2x^ry = 0}$ ***to the Bessel's Equation.*** Putting

$y = t^m$, so that $\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{1}{m}t^{1-m}\frac{dy}{dt}$

and $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{m}t^{1-m}\frac{dy}{dt}\right).\ \frac{1}{m}t^{1-m} = \frac{1}{m^2}t^{2-2m}\frac{d^2y}{dt^2} + \frac{1-m}{m^2}t^{1-2m}\frac{dy}{dt}$ in the above differential equation, we get

$\frac{1}{m^2}t^{2-m}\frac{d^2y}{dt^2} + \frac{1-m+cm}{m^2}\ t^{1-m}\frac{dy}{dt} + k^2t^{mr}y = 0$

Multiplying throughout by m^2/t^{1-m}, we get

$t\ \frac{d^2y}{dt^2} + (1-m+cm)\frac{dy}{dt} + (km)^2t^{mr+m-1}y = 0$

To reduce this equation to the equation at point 2. above, we set $mr + m - 1 = 1$

$i.e. m = 2/(r+1)$ and $a = 1 - m + cm = \frac{r+2c-1}{r+1}$.Thus we get the equation as

$t\ \frac{d^2y}{dt^2} + a\frac{dy}{dt} + (km)^2ty = 0$ which is similar to equation at point 2.

Hence its solution is

$y = x^{n/m}\left[c_1 J_n\left(km\,x^{1/m}\right) + c_2 J_{-n}\left(km\,x^{1/m}\right)\right],\ n$ is non-integral. or

$y = x^{n/m}\left[c_1 J_n\left(km\,x^{1/m}\right) + c_2 Y_n\left(km\,x^{1/m}\right)\right],\ n$ is integral.

ORTHOGONALITY OF BESSEL FUNCTIONS

Prove that $\int_0^1 xJ_n(\alpha x)J_n(\beta x)dx = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2}\left(J_{n+1}(\alpha)\right)^2, & \alpha = \beta \end{cases}$

where α, β are roots of $J_n(x) = 0$.

Proof: Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$ are the solutions of the following differential equations

$$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad (1)$$

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \qquad (2)$$

Multiplying equation (1) by v/x and equation (2) by u/x and then on subtracting, we get

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)xuv = 0$$

$$\Rightarrow \quad \frac{d}{dx}[x(u'v - uv')] = (\beta^2 - \alpha^2)xuv \qquad (3)$$

Now, integrating both sides of equation (3) within the limits 0 to 1, we get

$$(\beta^2 - \alpha^2)\int_0^1 x\,uv\,dx = [x(u'v - uv')]_0^1 = u'v - uv' \qquad (4)$$

Since $u = J_n(\alpha x)$ and $v = J_n(\beta x)$

$\therefore \quad u' = \alpha\, J_n'(\alpha x)$ and $v' = \beta\, J_n'(\beta x)$

Substituting these values in equation (4), we get

$$\int_0^1 x\,J_n(\alpha x)J_n(\beta x)\,dx = \frac{\alpha\, J_n'(\alpha)J_n(\beta) - \beta\, J_n'(\beta)J_n(\alpha)}{(\beta^2 - \alpha^2)} \qquad (5)$$

Case I: $\boldsymbol{\alpha \neq \beta}$

Since α, β are roots of $J_n(x) = 0$, so we have $J_n(\alpha) = J_n(\beta) = 0$. Thus equation (5) results in

$$\int_0^1 x\,J_n(\alpha x)J_n(\beta x)\,dx = 0 \qquad (6)$$

Case II: $\boldsymbol{\alpha = \beta}$

In this case RHS of (5) becomes 0/0 form. So to get its value, apply L'Hospital Rule, by taking α as constant and β as variable approaching to α, we get

$$\mathrm{Lim}_{\beta\to\alpha}\int_0^1 x\,J_n(\alpha x)J_n(\beta x)\,dx = \mathrm{Lim}_{\beta\to\alpha}\frac{\alpha\, J_n'(\alpha)J_n(\beta)}{(\beta^2 - \alpha^2)}\left(\frac{0}{0}\right) \qquad \text{or}$$

$$\mathrm{Lim}_{\beta\to\alpha}\int_0^1 x\,J_n^2(\alpha x)dx = \mathrm{lim}_{\beta\to\alpha}\frac{\alpha\, J_n'(\alpha)J_n'(\beta)}{2\beta}$$

$$= \frac{1}{2}(J_n'(\alpha))^2$$

$$= \frac{1}{2}(J_{n+1}(\alpha))^2 (using\ J_n' = -J_{n+1}) \qquad (7)$$

The relations (6) and (7) are known as Orthogonality relations of Bessel functions.

FOURIER BESSEL EXPANSION

If $f(x)$ is a continuous function having finite number of oscillations in the interval $(0,\ a)$, then we can write

$$f(x) = \sum_{j=1}^{\infty} c_j\ J_n(\alpha_j x) = c_1\ J_n(\alpha_1 x) + c_2\ J_n(\alpha_2 x) + \cdots + c_n\ J_n(\alpha_n x) + \cdots (1)$$

where $\alpha_1, \alpha_2, \cdots$ are the positive roots of $J_n(x) = 0$.

To determine the coefficients c_n, multiply both sides of (1) by $xJ_n(\alpha_n x)$ and integrating within the limits 0 to *a*, we get

$$\int_0^a x\, f(x)\, J_n(\alpha_n x)\, dx = c_n \int_0^a x\, {J_n}^2(\alpha_n x)\, dx = c_n\, \frac{a^2}{2}\, {J_{n+1}}^2(a\, \alpha_n)$$

$$\Rightarrow \quad c_n = \frac{2}{a^2 {J_{n+1}}^2(a\, \alpha_n)} \int_0^a x\, f(x)\, J_n(\alpha_n x)\, dx$$

The relation (1) is called Fourier Bessel Expansion of *f*(*x*).

BER AND BEI FUNCTIONS

The differential equation generally encountered in the field of electrical engineering for finding the distribution of alternating currents in wires of circular cross section is as follows:

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - i\, xy = 0 \qquad (1)$$

which is the special case of first form of differential equation reducible to Bessel equation with $n = 0$ and $k^2 = -i$, so that $k = \sqrt{-i} = i\sqrt{i} = i^{\frac{3}{2}}$(***Refer Art 18.9***).

Thus, the general solution of differential equation (1) is given by

$$y = c_1 J_0\left(i^{\frac{3}{2}}\, x\right) + c_2 Y_0\left(i^{\frac{3}{2}}\, x\right)$$

Now $\quad J_0\left(i^{\frac{3}{2}}\, x\right) = 1 - \frac{i^3 x^2}{2^2} + \frac{i^6 x^4}{(2!)^2\, 2^4} - \frac{i^9 x^6}{(3!)^2\, 2^6} + \frac{i^{12} x^8}{(4!)^2\, 2^8} - \cdots$

$$= \left[1 - \frac{x^4}{2^2.4^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \cdots\right]$$

$$+ i\left[\frac{x^2}{2^2} - \frac{x^6}{2^2.4^2.6^2} + \frac{x^{10}}{2^2.4^2.6^2.8^2.10^2} - \cdots\right] \qquad (2)$$

which is complex for *x* is real.

The series in the brackets of (2) is defined as

$$ber\ (x) = 1 - \frac{x^4}{2^2.4^2} + \frac{x^8}{2^2.4^2.6^2.8^2} - \cdots$$

$$= 1 + \sum_{m=1}^{\infty} (-1)^m . \frac{x^{4m}}{2^2.4^2.6^2 \cdots (4m)^2}$$

and $\quad bei\ (x) = \frac{x^2}{2^2} - \frac{x^6}{2^2.4^2.6^2} + \frac{x^{10}}{2^2.4^2.6^2.8^2.10^2} - \cdots$

$$= -\sum_{m=1}^{\infty} (-1)^m . \frac{x^{4m-2}}{2^2.4^2.6^2 \cdots (4m-2)^2}$$

where *ber* stands for Bessel real and *bei* for Bessel imaginary.

Thus we have $J_0\left(i^{\frac{3}{2}}x\right) = ber\,(x) + i\,bei(x)$

Similarly, decomposing $Y_0\left(i^{\frac{3}{2}}x\right)$ into real and imaginary parts, we obtain another two functions known as *ker* (*x*) and *kei* (*x*).

Properties of ber and bei functions

1. $\frac{d}{dx}\left[x.\frac{d}{dx}\,ber\,(x)\right] = -x\,bei\,(x)$
2. $\frac{d}{dx}\left[x.\frac{d}{dx}\,bei\,(x)\right] = -x\,ber\,(x)$

Example 17: Solve $\boldsymbol{y'' + \frac{y'}{x} + \left(1 - \frac{1}{9x^2}\right)y = 0}$

Solution: $y'' + \frac{y'}{x} + \left(1 - \frac{1}{9x^2}\right)y = 0$

$\Rightarrow$ $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$

Comparing with Bessel's equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

We find $n = \frac{1}{3}$

$\therefore$ The solution of the given equation is $y = c_1J_{\frac{1}{3}}(x) + c_2Y_{\frac{1}{3}}(x)$

Example 18: Solve $\boldsymbol{y'' + \frac{y'}{x} + \left(1 - \frac{1}{6.25\,x^2}\right)y = 0}$

Solution: $y'' + \frac{y'}{x} + \left(1 - \frac{1}{6.25\,x^2}\right)y = 0$

$\Rightarrow$ $y'' + \frac{y'}{x} + \left(1 - \frac{100}{625\,x^2}\right)y = 0$

Comparing with the Bessel's equation, we find $n = \frac{10}{25} = \frac{2}{5}$

$\therefore$ The solution of the given equation is $y = c_1J_{\frac{2}{5}}(x) + c_2Y_{\frac{2}{5}}(x)$

Example 19: Solve $\boldsymbol{xy'' + y' + \frac{1}{4}y = 0}$

Solution: Let $t = x^{\frac{1}{m}}$, so that

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{1}{m}x^{\frac{1}{m}-1}.\frac{dy}{dt} = \frac{1}{m}(t)^{1-m}.\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{1}{m}t^{m-1}.\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= \left[\frac{1}{m}.(1-m)t^{-m}.\frac{dy}{dt} + \frac{1}{m}t^{1-m}\frac{d^2y}{dt^2}\right] \times \frac{1}{m}t^{1-m}$$

$$= \frac{1}{m^2}(1-m)t^{1-2m}\frac{dy}{dt} + \frac{1}{m^2}t^{2-2m}\frac{d^2y}{dt^2}$$

$\therefore$ $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{1}{4}y$

$$= \frac{t^m}{m^2}\left[(1-m)t^{1-2m}\frac{dy}{dt} + t^{\,2-2m}\frac{d^2y}{dt^2}\right] + \frac{1}{m}t^{1-m}\frac{dy}{dt} + \frac{1}{4}y = 0$$

$\Rightarrow$ $$\frac{1}{m^2}t^{2-m}\frac{d^2y}{dt^2} + \frac{1-m}{m^2}t^{1-m}\frac{dy}{dt} + \frac{1}{m}t^{1-m}\frac{dy}{dt} + \frac{1}{4}y = 0$$

$\Rightarrow$

$$t^2\frac{d^2y}{dt^2} + (1-m+m)t\frac{dy}{dt} + \frac{1}{4}m^2t^m y = 0$$

$\Rightarrow t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \frac{1}{4}m^2t^m y = 0$

Comparing with

$$x\frac{d^2y}{dx^2} + a\frac{dy}{dx} + k^2ny = 0$$

We get $a = 1, \quad k^2 = \frac{m^2}{4}, \quad m - 1 = 1$ it implies $m = 2$

i.e. $k^2 = 1$ and $n = \frac{1-a}{2} = 0$

$\therefore$ The solution of the given equation is

$y = c_1J_0(t) + c_2Y_0(t) = c_1J_0\left(\sqrt{x}\right) + c_2Y_0\left(\sqrt{x}\right)$

Example 20: Solve $\boldsymbol{xy'' + 2y' + \frac{1}{2}xy = 0}$

Solution: Let $y = x^n z$ so that

$$\frac{dy}{dx} = x^n\frac{dz}{dx} + nx^{n-1}z$$

$$\frac{d^2y}{dx^2} = x^n\frac{d^2z}{dx^2} + 2nx^{n-1}\frac{dz}{dx} + n(n-1)x^{n-2}z$$

$\therefore xy'' + 2y' + \frac{1}{2}xy = 0$

$\Rightarrow$ $$x^{n+1}\frac{d^2z}{dx^2} + (2n+2)x^n\frac{dz}{dx} + \left[\{n(n-1) + 2n\}x^{n-1} + \frac{1}{2}x^{n+1}z\right] = 0$$

$\Rightarrow$ $$x^2\frac{d^2z}{dx^2} + 2(n+1)x\frac{dz}{dx} + \left\{n(n+1) + \frac{1}{2}x^2\right\}z = 0$$

Taking $2(n+1) = 1$ i.e. $n = -\frac{1}{2}$

$\Rightarrow$ $$x^2\frac{d^2z}{dx^2} + x\frac{dz}{dx} + \left(\frac{1}{2}x^2 - \frac{1}{4}\right)z = 0$$

$\Rightarrow z = c_1J_{\frac{1}{2}}\left(\sqrt{\frac{1}{2}}\,x + c_2Y_{\frac{1}{2}}\left(\sqrt{\frac{1}{2}}\,\,x\right)\right)$

$\Rightarrow y = x^{-\frac{1}{2}}\left\{c_1J_{\frac{1}{2}}\left(\frac{x}{\sqrt{2}}\right) + c_2Y_{\frac{1}{2}}\left(\frac{x}{\sqrt{2}}\right)\right\}$

Example 21: Solve $\boldsymbol{xy'' + y = 0}$ (1)

Solution: Let $t = x^{\frac{1}{m}}$, so that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{m}t^{1-m}\frac{dy}{dt}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{1}{m}t^{1-m}\frac{dy}{dt}\right] \times \frac{dt}{dx}$$

$$= \left\{\frac{1}{m}t^{1-m}\frac{d^2y}{dt^2} + \frac{1}{m}(1-m)t^{-m}.\frac{dy}{dt}\right\}\frac{1}{m}t^{1-m}$$

$\therefore xy'' + y = 0$

$$\Rightarrow \quad t^m\left\{\frac{1}{m^2}t^{2-2m}\frac{d^2y}{dt^2} + \frac{1}{m^2}(1-m)t^{1-2m}\frac{dy}{dt}\right\} + y = 0$$

$$\Rightarrow \quad t^{2-m}\frac{d^2y}{dt^2} + (1-m)t^{1-m}\frac{dy}{dt} + m^2y = 0$$

$$\Rightarrow \quad t\frac{d^2y}{dt^2} + (1-m)\frac{dy}{dt} + m^2t^{m-1}y = 0 \qquad (2)$$

Comparing both

$$xy'' + ay' + k^2xy = 0$$

We will have

$a = 1 - m,\ k = m$ and $m - 1 = 1$

i.e. $m = 2,\ k = 2$ and $a = 1 - 2 = -1$

$\therefore n = \frac{1-a}{2} = \frac{1+1}{2} = 1$

Hence the solution of the equation (2) will be

$$y = t\{c_1J_1(2t) + c_2Y_1(2t)\}$$

$$\Rightarrow y = x^{\frac{1}{2}}\{c_1J_1(2\sqrt{x}) + c_2Y_1(2\sqrt{x})\}$$

Example 22: Solve $\boldsymbol{y'' + \left(9x - \frac{20}{x^2}\right)y = 0}$ (1)

Solution: Let $t = x^{\frac{1}{m}}$ or $x = t^m$, so that

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{m}t^{1-m}\frac{dy}{dt}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{1}{m}t^{1-m}\frac{dy}{dt}\right] \times \frac{dt}{dx}$$

$$= \left\{\frac{1}{m}t^{1-m}\frac{d^2y}{dt^2} + \frac{1}{m}(1-m)t^{-m}.\frac{dy}{dt}\right\}\frac{1}{m}t^{1-m}$$

$$\Rightarrow \quad \frac{1}{m^2}t^{2-2m}\frac{d^2y}{dt^2} + \frac{1}{m^2}(1-m)t^{1-2m}\frac{dy}{dt} + \left(9t^m - \frac{20}{t^{2m}}\right)y = 0$$

$$\Rightarrow \quad t^2\frac{d^2y}{dt^2} + (1-m)t\frac{dy}{dt} + m^2\{9t^{3m} - 20\}y = 0$$

$$\Rightarrow \quad t^2\frac{d^2y}{dt^2} + (1-m)t\frac{dy}{dt} + (9m^2t^{3m} - 20m^2)y = 0$$

Taking $3m = 2$ i.e. $m = \frac{2}{3}$, we will have

$$t^2\frac{d^2y}{dt^2} + \frac{1}{3}t\frac{dy}{dt} + \left(4t^2 - \frac{80}{9}\right)y = 0 \qquad (2)$$

Now let $y = t^nz(t)$, so that

$$\frac{dy}{dt} = t^n \frac{dz}{dt} + nt^{n-1}z,$$

$$\frac{d^2y}{dt^2} = t^n \frac{d^2z}{dt^2} + 2nt^{n-1}\frac{dz}{dt} + n(n-1)t^{n-2}z$$

Substituting these in (2), we get

$$t^{n+2}\frac{d^2z}{dt^2} + \left\{2n + \frac{1}{3}\right\} t^{n+1}\frac{dz}{dt} + \left[\left\{n(n-1) + \frac{1}{3}n - \frac{80}{9}\right\} t^n + 4t^{n+2}\right] z = 0 \qquad (3)$$

Now for $2n + \frac{1}{3} = 1$, $\quad n = \frac{1}{3}$

and $\quad n(n-1) + \frac{1}{3}n - \frac{80}{9} = \frac{1}{3} \times -\frac{2}{3} + \frac{1}{3} \times \frac{1}{3} - \frac{80}{9} = -\frac{81}{9} = -9$

$\therefore$ Dividing (3) by t^n and substituting for n, we will have

$$t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} + \{4t^2 - 9\}z = 0 \qquad (4)$$

The solution of (4) is

$$z = c_1J_3(2t) + c_2Y_3(2t)$$

$$\Rightarrow y = t^{\frac{1}{3}}[c_1J_3(2t) + c_2Y_3(2t)]$$

$$\Rightarrow y = \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} \left[c_1J_3\left(2x^{\frac{3}{2}}\right) + c_2Y_3\left(2x^{\frac{3}{2}}\right)\right]$$

$$= x^{\frac{1}{2}} \left[c_1J_3\left(2x^{\frac{3}{2}}\right) + c_2Y_3\left(2x^{\frac{3}{2}}\right)\right]$$

Example 23: Show that

(i) $\boldsymbol{x^nJ_n(x)}$ **is a solution of the equation** $\boldsymbol{xy'' + (1-2n)y' + xy = 0}$.

(ii) $\boldsymbol{x^{-n}J_n(x)}$ **is the solution of the equation** $\boldsymbol{xy'' + (1+2n)y' + xy = 0}$

Solution: Let $\quad y = x^nJ_n(x)$

$$\therefore \frac{dy}{dx} = x^nJ_n'(x) + nx^{n-1}J_n(x)$$

and $\quad \frac{d^2y}{dx^2} = x^nJ_n''(x) + 2nx^{n-1}J_n'(x) + n(n-1)x^{n-2}J_n(x)$

$$\therefore xy'' + (1-2n)y' + xy$$

$$= x^{n+1}J_n''(x) + 2nx^nJ_n'(x) + n(n-1)x^{n-1}J_n(x)$$
$$+(1-2n)\{x^nJ_n'(x) + nx^{n-1}J_n(x) + x^{n+1}J_n(x)\}$$

$$= x^{n+1}J_n''(x) + x^nJ_n'(x)\{2n + 1 - 2n\}$$

$$+\{[n(n-1) + n(1-2n)]x^{n-1} + x^{n+1}\}J_n(x)$$

$$= x^{n-1}[x^2J_n''(x) + J_n'(x) + (x^2 - n^2)J_n(x)] = 0$$

As $J_n(x)$ is the Bessel function and is a solution of $x^2y'' + y' + (x^2 - n^2)y = 0$

Hence, $x^nJ_n(x)$ satisfy the given equation and therefore is a solution of it.

Example 24: Show under the transformations $\boldsymbol{y = \frac{u}{\sqrt{x}}}$ **Bessel's equation becomes** $\boldsymbol{u'' +}$

$\left\{1+\frac{1-4n^2}{4x^2}\right\}u=0$; **Hence find the solution of this equation.**

Solution: We know that the Bessel's equation is

$$x^2y'' + xy' + (x^2 - n^2)y = 0 \qquad (1)$$

Taking $y = \frac{u}{\sqrt{x}}$

$\Rightarrow$ $$y' = \frac{1}{\sqrt{x}}u' + \left(-\frac{1}{2}\right)x^{-\frac{3}{2}}u,$$

and $$y'' = \frac{1}{\sqrt{x}}u'' + 2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}u' + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}u$$

Substituting these into (1), we get

$$x^2\left\{\frac{1}{\sqrt{x}}u'' - x^{-\frac{3}{2}}u' + \frac{3}{4}x^{-\frac{5}{2}}u\right\} + x\left\{\frac{1}{\sqrt{x}}u' + \left(-\frac{1}{2}\right)x^{-\frac{3}{2}}u\right\} + (x^2-n^2)\frac{u}{\sqrt{x}} = 0$$

$\Rightarrow$ $$x^{\frac{3}{2}}u'' + \left\{-x^{\frac{1}{2}} + x^{\frac{1}{2}}\right\}u' + \left\{\frac{3}{4}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}} + x^{\frac{3}{2}} - \frac{n^2}{\sqrt{x}}\right\}u = 0$$

$\Rightarrow$ $$x^{\frac{3}{2}}u'' + \left\{x^{\frac{3}{2}} + \frac{3-2-4n^2}{4\sqrt{x}}\right\}u = 0$$

$\Rightarrow$ $$u'' + \left\{1 + \frac{1-4n^2}{4x^2}\right\}u = 0 \qquad (2)$$

Hence the Bessel's equation (1) becomes (2) as desired.

Now the solution of (1) is $y = c_1J_n(x) + c_2Y_n(x)$ (3)

$\Rightarrow$ $$\frac{u}{\sqrt{x}} = c_1J_n(x) + c_2Y_n(x)$$

$\Rightarrow$ $$u = \sqrt{x}\{c_1J_n(x) + c_2Y_n(x)\}$$

Example 25: By the use of the substitution $y = \frac{u}{\sqrt{x}}$ so that the solution of the equation
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0$$
can be written in the form
$$y = c_1\frac{\sin x}{\sqrt{x}} + c_2\frac{\cos x}{\sqrt{x}}.$$

Solution: Taking

$$y = \frac{u}{\sqrt{x}}$$

$\Rightarrow$ $$\frac{dy}{dx} = \frac{1}{\sqrt{x}}\frac{du}{dx} - \frac{1}{2}x^{-\frac{3}{2}}u$$

and

$$\frac{d^2y}{dx^2} = x^{-\frac{1}{2}}\frac{d^2u}{dx^2} - x^{-\frac{3}{2}}\frac{dy}{dx} + \frac{3}{4}x^{-\frac{5}{2}}u$$

Substituting these in the given equation, we get

$$x^2\left\{x^{-\frac{1}{2}}\frac{d^2u}{dx^2}-x^{-\frac{3}{2}}\frac{du}{dx}+\frac{3}{4}x^{-\frac{5}{2}}u\right\}+x\left\{x^{-\frac{1}{2}}\frac{du}{dx}-\frac{1}{2}x^{-\frac{3}{2}}u\right\}+\left(x^2-\frac{1}{4}\right)x^{-\frac{1}{2}}u=0$$

$\Rightarrow$ $$x^{\frac{3}{2}}\frac{d^2u}{dx^2}-x^{\frac{1}{2}}\frac{du}{dx}+\frac{3}{4}x^{-\frac{1}{2}}u+x^{\frac{1}{2}}\frac{du}{dx}-\frac{1}{2}x^{-\frac{1}{2}}u+\left(x^{\frac{3}{2}}-\frac{1}{4}x^{-\frac{1}{2}}\right)u=0$$

$\Rightarrow$ $x^{\frac{3}{2}}\frac{d^2y}{dx^2}+x^{\frac{3}{2}}u=0$ It implies $\frac{d^2y}{dx^2}+u=0$

Its Auxiliary equation is $D^2+1=0$ it implies $D=\pm i$

$\therefore$ $u(x)=c_1\cos x+c_2\sin x$

Hence $y=\frac{u}{\sqrt{x}}=c_1\frac{\cos x}{\sqrt{x}}+c_2\frac{\sin x}{\sqrt{x}}$

Example 26: Show that

$$\int_0^p x\left(ber^2x+bei^2x\right)dx=p(ber\,p.bei'p-bei\,p.ber'\,p)$$

Solution: We know $ber\,x=1+\sum_{m=1}^{\infty}(-1)^m\frac{x^{4m}}{2^2.4^2.6^2....(4m)^2}$ and

$bei\,x=-\sum_{m=1}^{\infty}(-1)^m\frac{x^{4m-2}}{2^2.4^2.6^2......(4m-2)^2}$

$\Rightarrow$ $\frac{d}{dx}(x\,bei'\,x)=x\,ber\,x$

$\therefore$ $\int_0^p x\,(ber^2\,x+bei^2\,x)dx=\int_0^p\{(x\,ber\,x).ber\,x+(x\,bei\,x)bei\,x\}dx$

$=\int_0^p\left\{\frac{d}{dx}(x\,bei'\,x).ber\,x-\frac{d}{dx}(x\,ber'\,x)bei\,x\right\}dx$

$=[ber\,x.\{x\,bei'\,x\}-\int ber'\,x(x\,bei'\,x\,dx)-bei\,x\,\{x\,bei'\,x+bei'xx\,ber'xdx0p$

$=p(ber\,p.bei'p-bei\,p.ber'\,p)$ hence proved

Example 27: If $a_1,a_2,a_3,\ldots\ldots a_n$ are the positive roots of $J_0(x)=0$, prove that

(i) $\frac{1}{2}=\sum_{n=1}^{\infty}\frac{J_0(\alpha_n x)}{\alpha_nJ_1(\alpha_n)}$**(ii)** $x^2=2\sum_{n=1}^{\infty}\frac{\alpha_n{}^2-4}{\alpha_n{}^3J_1(\alpha_n)}J_0(\alpha_n x)$

Solutions:(i)

Let the Fourier Bessel expression of

$\frac{1}{2}$ is $\frac{1}{2}=\sum_{n=1}^{\infty}c_nJ_0(\alpha_n\,x)$

and integrating with respect to $'x'$ from 0 to 1, we get

$\int_0^1\frac{1}{2}xJ_0(\alpha_n\,x)\,dx=c_n\int_0^1 xJ_0{}^2(\alpha_n\,x)dx=c_n\frac{1}{2}[J_1(\alpha_n)]^2$

$\Rightarrow c_n\frac{1}{2}J_1{}^2(\alpha_n)=\frac{1}{2}\int_0^1 x\,J_0(\alpha_n\,x)$

Let $\alpha_nx=t$ it implies $dx=\frac{dt}{\alpha_n}$

$x\to(0,1)$ It implies $t\to(0\;to\;\alpha_n)$

$$= \frac{1}{2}\int_0^{\alpha_n} \frac{t}{\alpha_n} J_0(t)\frac{dt}{\alpha_n}$$

$$= \frac{1}{2\alpha_n{}^2}\int_0^{\alpha_n} t\,J_0(t)dt = \frac{1}{2\alpha_n{}^2}\int_0^{\alpha_n} \frac{d}{dt}\big(t\,J_1(t)\big)dt$$

$$= \frac{1}{2\alpha_n{}^2}[t\,J_1(t)]_0^{\alpha_n} = \frac{1}{2\alpha_n{}^2}[\alpha_n J_1(\alpha_n)]$$

$$\therefore c_n \frac{1}{2} J_1{}^2(\alpha_n) = \frac{1}{2\alpha_n} J_1(\alpha_n)$$

It implies $c_n = \frac{1}{\alpha_n J_1(\alpha_n)}$ Hence $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{\alpha_n J_1(\alpha_n)} J_0(\alpha_n\, x)$

(ii) Let the Fourier-Bessel expansion of x^2 is $x^2 = \sum_{n=1}^{\infty} c_n J_0(\alpha_n\, x)$ and integrating from 0 to 1,

we get $\int_0^1 x^3 J_0(\alpha_n\, x) = c_n \int x\,J_0{}^2(\alpha_n\, x)dx$

$\Rightarrow\ c_n \frac{1}{2} J_1{}^2(\alpha_n) = \int_0^{\alpha_n} \frac{t^3}{\alpha_n{}^3} J_0(t)\frac{1}{\alpha_n}dt$, if $\alpha_n x = t$ it implies $dx = \frac{dt}{\alpha_n}$

$$= \frac{1}{\alpha_n{}^4}\int_0^{\alpha_n} t^2 \frac{d}{dt}\big(t\,J_1(t)\big)\,dt$$

$$= \frac{1}{\alpha_n{}^4}[t^2 . tJ_1(t) - \int 2t\,.\,tJ_1(t)dt]_0^{\alpha_n}$$

$$= \frac{1}{\alpha_n{}^4}\left[t^3 J_1(t) - 2\int \frac{d}{dt}(t^2 J_2(t)dt)\right]_0^{\alpha_n}$$

$$= \frac{1}{\alpha_n{}^4}\left[t^3 J_1(t) - 2\int \frac{d}{dt}\big(t^2 J_2(t)\big)dt\right]_0^{\alpha_n}$$

$$= \frac{1}{\alpha_n{}^4}[t^3 J_1(t) - 2t^2 J_2(t)]_0^{\alpha_n}$$

$$= \frac{1}{\alpha_n{}^4}[\alpha_n{}^3 J_1(\alpha_n) - 2\alpha_n{}^2 J_2(\alpha_n)]$$

$$= \frac{1}{\alpha_n{}^2}[\alpha_n J_1(\alpha_n) - 2J_2(\alpha_n)]$$

$$= \frac{1}{\alpha_n{}^2}\left[\alpha_n J_1(\alpha_n) - 2\left\{\frac{2}{\alpha_n} J_1(\alpha_n) - J_0(\alpha_n)\right\}\right]$$

$$= \frac{1}{\alpha_n{}^2}\left[\left(\frac{\alpha_n{}^2-4}{\alpha_n}\right) J_1(\alpha_n) - J_0(\alpha_n)\right]$$

$$= \left(\frac{\alpha_n{}^2-4}{\alpha_n{}^3}\right) J_1(\alpha_n) \quad \text{as} \quad J_0(\alpha_n) = 0$$

$$\therefore \quad c_n = \frac{2}{J_1(\alpha_n)}\left(\frac{\alpha_n{}^2-4}{\alpha_n{}^3}\right)$$

Hence $x^2 = 2\sum\left(\frac{\alpha_n{}^2-4}{\alpha_n{}^3 J_1(\alpha_n)}\right) J_0(\alpha_n\, x)$

Example 28: Expand $f(x) = x^2$ in the interval $0 < x < 3$ in terms of function $J_1(\alpha_n\, x)$ where α_n are determined by $J_1(3\alpha) = 0$.

Solution: Let the Fourier-Bessel expansion of $f(x) = x^2$ is

$x^2 = \sum_{n=1}^{\infty} c_n\, J_1(\alpha_n\, x)$,

multiplying both sides by $xJ_1(\alpha_n\, x)$ and integrating from 0 to 3, we get

$$\int_0^3 x^4 J_1(\alpha_n x)dx = c_n \int_0^3 xJ_1(\alpha_n x)dx$$

Let $x = 3t$ so that $dx = 3dt$

$$8\int_0^t t^4 J_1(3\alpha_n t)\ 3dt = c_n \int_0^1 3t\, J_1{}^2(3\alpha_n t)3dt$$

$\therefore$ $$c_n \int_0^1 tJ_1{}^2\,(3\alpha_n t)dt = 27\int_0^1 t^4 J_1(3\alpha_n t)dt$$

$$c_n \frac{1}{2} J_2{}^2(3\alpha_n) = 27\int_0^{3\alpha_n} \frac{z^4}{81\alpha_n{}^4} J_1(z)\frac{dz}{3\alpha_n}$$ (where $3\alpha_n t = z$ and $dt = \frac{dz}{3\alpha_n}$)

$$= \frac{1}{9\alpha_n{}^5}\int_0^{3\alpha_n} z^4 J_1(z)dz$$

$$= \frac{1}{9\alpha_n{}^5}\int_0^{3\alpha_n} z^2 \frac{d}{dz}\left(z^2 J_2(z)\right)dz$$

$$= \frac{1}{9\alpha_n{}^5}\left[z^2.z^2 J_2(z) - \int 2z.z^2 J_2(z)dz\right]_0^{3\alpha_n}$$

$$= \frac{1}{9\alpha_n{}^5}\left[z^4 J_2(z) - 2\int \frac{d}{dz}(z^3/\ {}_3(z)dz)\right]_0^{3\alpha_n}$$

$$= \frac{1}{9\alpha_n{}^5}\left[z^4 J_2(z) - 2z^3 J_3(z)\right]_0^{3\alpha_n}$$

$$= \frac{1}{9\alpha_n{}^5}\left[81\alpha_n{}^4 J_2(3\alpha_n) - 2\times 27\alpha_n{}^3 J_3(3\alpha_n)\right]$$

$$= \frac{1}{\alpha_n{}^2}\left[9\alpha_n J_2(3\alpha_n) - 2J_3(3\alpha_n)\right]$$

$\therefore$ $$c_n = \frac{6}{\alpha_n{}^2 J_2{}^2(3\alpha_n)}\left[3\alpha_n J_2(3\alpha_n) - 2J_3(3\alpha_n)\right]$$

Hence $$x^3 = 6\sum_{n=1}^{\infty} \frac{3\alpha_n J_2(3\alpha_n) - 2J_3(3\alpha_n)}{\alpha_n{}^2 J_2{}^2(3\alpha_n)} J_1(\alpha_n x)$$

ASSIGNMENT

Solve the differential equations:

(i) $y'' + \frac{y'}{x} + \left(8 - \frac{1}{x^2}\right)y = 0$

(ii) $4y'' + 9xy = 0$

(iii) $x^2 y'' - xy' + 4x^2 y = 0$

2. If $\alpha_1,\ \alpha_2, \dots,\ \alpha_n$ are the positive roots of $J_0(x) = 0$, show that

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{J_0(\alpha_n x)}{\alpha_n J_1(\alpha_n)}$$

3. Expand $f(x) = x^2$ in the interval $0 < x < 2$ in terms of $J_2(\alpha_n x)$, where $\alpha_n$are determined by $J_2(\alpha_n) = 0$.

4. Prove that

(i) $\frac{d}{dx}\left[x.\frac{d}{dx}\ ber\ (x)\right] = -x\ bei\ (x)$

(ii) $\frac{d}{dx}\left[x.\frac{d}{dx}\ bei\ (x)\right] = -x\ ber\ (x)$

ANSWERS

1.(i) $y = C_1 J_1(2\sqrt{2x}) + C_2 Y_{-1}(2\sqrt{2x})$

(ii) $y = \sqrt{x}\left(C_1 J_{\frac{1}{3}}\left(x^{\frac{3}{2}}\right) + C_2 Y_{-\frac{1}{3}}\left(x^{\frac{3}{2}}\right)\right)$

(iii) $y = x\left(C_1 J_1(2x) + C_2 Y_1(2x)\right)$

$x^2 = 4\sum_{n=1}^{\infty} \frac{J_2(\alpha_n x)}{\alpha_n J_3(2\alpha_n)}$ 3.

LEGENDRE'S EQUATION

Legendre's equation is one of the important differential equations occurring in applied mathematics, particularly in boundary value problems for spheres. It is given as

$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ (1) where n is given real number. In most applications, n takes integral values.

The singularities of this equation are $x = \pm 1$.

Substituting $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots (a_0 \neq 0)$ in (1), we get

$a_0\,(m)(m-1)x^{m-2} + c\;\;(m+1)mx^{m-1} + \cdots$

$+ [a_{r+2}(\;\;\;+ r + 2)(m+r+1) - \{(m+r)(m+r+1) - n(n+1)\}a_r]x^{m+r}$

$+ \cdots = 0$

Equating to zero the co-efficient lowest powers of x, *i.e*of x^{m-2}, we get

$a_0\,(m)(m-1) = 0 \quad \Rightarrow m = 0, 1 \qquad (a_0 \neq 0)$

Equating to zero the co-efficient of x^{m-1} and x^{m+r}, we get

$a_1(m+1)m = 0$ (2)

$a_{r+2}(m+r+2)(m+r+1) - \{(m+r)(m+r+1) - n(n+1)\}a_r = 0$ (3)

When $m = 0$, (2) is satisfied and therefore $a_1 \neq 0$. Then (3) for $r = 0, 1, 2, 3$ ... gives

$a_2 = -\frac{n(n+1)}{2!}a_0; \quad a_3 = -\frac{(n-1)(n+2)}{3!}a_1;$

$a_4 = -\frac{(n-2)(n+3)}{4\,.3}a_2 = \frac{n(n-2)(n+1)(n+3)}{4!}a_0;$

$a_5 = -\frac{(n-3)(n+4)}{5\,.\,4}a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1; \;\; etc.$

Therefore two independent solutions of (1) for $m = 0$ are as follows:

$y_1 = a_0\left\{1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \cdots\right\}$ (4)

$y_2 = a_1\left\{x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \cdots\right\}$ (5)

When $m = 1$, (2) gives that $a_1 = 0$. Therefore (3) gives

$a_3 = a_5 = a_7 = 0$

$a_2 = -\frac{n(n+1)}{2!}a_0; \quad a_4 = \frac{n(n-2)(n+1)(n+3)}{4!}a_0; \;\; etc$

Thus for $m = 1$, we get the solution (5) again. Hence the general solution of (1) is given by

$y = y_1 + y_2$.

Further, it is worth to note that if n is positive even integer, then (4) terminates at the term containing x^n and y_1 becomes a polynomial of degree n. Similarly, if n is positive odd integer, then y_2 becomes a polynomial of degree n. Thus, whenever n is a positive integer (even or odd), the general solution of (1) always contains a polynomial of degree n and an infinite series.

These polynomial solutions, with a_0 and a_1 chosen properly so that the value of the polynomial becomes one at $x = 1$, are called ***Legendre's Polynomials of degree n and is denoted by*** $\boldsymbol{P_n(x)}$. The infinite series with a_0 and a_1 chosen properly is called ***Legendre's Function of second kind and is denoted by*** $\boldsymbol{Q_n(x)}$.

RODRIGUE'S FORMULA

Another presentation of Legendre's Polynomials is given by

$$\boldsymbol{P_n(x) = \frac{1}{n!\ 2^n} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n}$$ **(1)** ***is known as Rodrigue's Formula.***

Proof: Let $v = (x^2 - 1)^n$, then $v_1 = \frac{dv}{dx} = 2nx\,(x^2 - 1)^{n-1}$

i.e. $(1 - x^2)v_1 + 2nx\,v = 0$ (2)

Differentiating (2), n+1 times by Leibnitz' theorem,

$$(1 - x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + \frac{1}{2!}(n+1)n(-2)v_n$$

$$+2n[xv_{n+1} + (n+1)v_n] = 0$$

or $$(1 - x^2)\frac{d^2(v_n)}{dx^2} - 2x\frac{d(v_n)}{dx} + n(n+1)(v_n) = 0$$

which is Legendre's Equation and cv_n is its solution. Also its finite series solution is $P_n(x)$.

$$\therefore P_n(x) = cv_n = c\,\frac{d^n}{dx^n}\,(x^2 - 1)^n \qquad (3)$$

Putting $x = 1$ in equation (3) for determining the value of the constant c, we get

$$1 = c\left[\frac{d^n}{dx^n}\,\{(x-1)^n(x+1)^n\}\right]_{x=1}$$

$$= c[n!\ (x+1)^n + terms\ with\ (x-1) and\ its\ powers]_{x=1}$$

$$= c.n!\,(2)^n,\ i.e.,\ c = \frac{1}{n!\,2^n}$$

Substituting the value of c in (3), we get eqution (1) which is known as ***Rodrigue's formula.***

LEGENDRE'S POLYNOMIALS

By Rodrigue's formula we have

$P_0(x) = 1,$ $P_1(x) = x,$

$P_2(x) = \frac{1}{2}(3x^2 - 1),$ $P_3(x) = \frac{1}{2}(5x^3 - 3x),$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$

$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$.

In general, $P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n\, r!(n-r)!(n-2r)!}\, x^{n-2r}$ where $N = \frac{1}{2}n$ or $\frac{1}{2}(n-1)$ according as n is even or odd.

This general expression for $P_n(x)$ in terms of sum of finite number of terms can be derived easily from Rodrigue's formula.

Example 29: Show that $P_n(-x) = (-1)^n P_n(x)$

Solution: $P_n(x) = \sum_{r=0}^{N}(-1)^2 \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} x^{n\pm 2r}$

Where $N = \frac{n}{2}$ or $\frac{n-1}{2}$

$\therefore$ Replacing x by $-x$, we will get

$$P_n(x) = \sum_{r=0}^{N}(-1)^r \frac{(2n-2r)!}{r!(n-2r)!(n-r)!}(-1)^{n-2r} x^{n-2r}$$

$$= (-1)^n \sum_{r=0}^{\infty}(-1)^r \frac{(2n-2r)!}{r!(n-r)!(n-2r)!} x^{n-2r}, \quad \text{as } (-1)^{2r} = 1$$

$$= (-1)^n P_n(x)$$

Example 30: Express the following in the Legendre Polynomials

(i) $\boldsymbol{5x^3 + x}$ **(ii)** $\boldsymbol{x^3 + 2x^2 - x - 3}$ **(iii)** $\boldsymbol{4x^3 - 2x^2 - 3x + 8}$

Solution: We know $P_n(x) = \frac{1}{n!\,2^n} D^n (x^2-1)^n$

$\therefore P_0(x) = 1 P_1(x) = x P_2(x) = \frac{1}{2}(3x^2 - 1) \quad P_3(x) = \frac{1}{2}(5x^3 - 3x)$

(i)

(ii)

(iii) $4x^3 - 2x^2 - 3x + 8 = \frac{4}{5}[2P_3(x) + 3P_1(x)] - \frac{2}{3}[2P_2(x) + P_0(x)] - 3P_1(x) + 8P_0(x)$
$= \frac{8}{5}P_3(x) - \frac{4}{3}P_2(x) - \frac{9}{5}P_1(x) + \frac{22}{3}P_0(x)$

GENERATING FUCTION FOR $P_n(x)$

To show that $\boldsymbol{(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)}$

Proof: We know that

$$(1-z)^{-\frac{1}{2}} = 1 + \frac{1}{2}z + \frac{\frac{1}{2}\cdot\frac{3}{2}}{2!} z^2 + \frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!} z^3 + \cdots$$
$$= 1 + \frac{2!}{(1!)^2\, 2^2} z + \frac{4!}{(2!)^2\, 2^4} z^2 + \frac{6!}{(3!)^2\, 2^6} z^3 + \cdots$$

$\therefore \quad \left(1 - t(2x - t)\right)^{-\frac{1}{2}} = 1 + \frac{2!}{(1!)^2\, 2^2}\left(t(2x - t)\right) + \frac{4!}{(2!)^2\, 2^4}\left(t(2x - t)\right)^2 + \cdots$

$+ \frac{(2n-2r)!}{((n-r)!)^2\, 2^{2n-2r}}\,(t(2x - t))^{n-r} + \cdots + \frac{(2n)!}{(n!)^2\, 2^{2n}}\,(t(2x - t))^n$ (1)

The term in t^n from the term containing $t^{n-r}\ (2x - t)^{n-r}$

$= \frac{(2n-2r)!}{((n-r)!)^2\, 2^{2n-2r}}\ t^{n-r}.n - r_{C_r}(-t)^r(2x)^{n-2r}$

$= \frac{(2n-2r)!}{((n-r)!)^2\, 2^{2n-2r}} \times \frac{(n-r)!}{r!(n-2r)!}\ (-1)^r t^n (2x)^{n-2r} = \frac{(-1)^r (2n-2r)!}{2^n\ r!\ (n-r)!(n-2r)!} t^n x^{n-2r}$

Collecting all terms in t^n which will occur in the term containing $t^n\ (2x - t)^n$ and the proceeding terms, we see that terms in t^n

$= \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n\ r!\ (n-r)!(n-2r)!} t^n x^{n-2r} = P_n(x) t^n$

where $N = \frac{1}{2}\ n\ \ or\ \ \frac{1}{2}(n - 1)$ according as n is even or odd.

Hence (1) can be written as $(1 - 2xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$, which is known as generating function of Legendre's Polynomials.

RECURRENCE RELATION FOR $P_n(x)$

I. $\boldsymbol{(n + 1)P_{n+1}(x) = (2n + 1)P_n(x) - nP_{n-1}(x)}$.

Proof: We have the generating functions

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \tag{1}$$

Differentiate partially w.r.t.*t*, we get

$$-\frac{1}{2}(1 - 2xt + t^2)^{-\frac{3}{2}}(-2x + 2t) = \sum_{n=0}^{\infty} P_n(x)\, nt^{n-1}$$

$$(1 - 2xt + t^2)^{-\frac{3}{2}}(x - t) = \sum_{n=0}^{\infty} P_n(x)\, nt^{n-1} \tag{2}$$

$$(1 - 2xt + t^2)^{-\frac{1}{2}}(x - t) = (1 - 2xt + t^2)\sum_{n=0}^{\infty} P_n(x)\, nt^{n-1}$$

$$(x - t)\sum_{n=0}^{\infty} P_n(x)\, t^{n-1} = (1 - 2xt + t^2)\sum_{n=0}^{\infty} P_n(x)\, nt^{n-1}$$

Comparing the coefficients of t^n from both sides, we get

$x\, P_n(x) - P_{n-1}(x) = (n + 1)P_{n+1}(x) - 2xnP_n(x) + (n - 1)P_{n-1}(x)$

$(n + 1)\ P_{n+1}(x) = (2n + 1)x\ P_n(x) - nP_{n-1}(x)$

II. $\boldsymbol{n\ P_n(x) = xP_n{}'(x) - P'{}_{n-1}(x)}$.

Proof: Differentiating (1) partially w.r.t x, we obtain

$-\frac{1}{2}(1 - 2xt + t^2)^{-\frac{3}{2}}(-2t) = \sum_{n=0}^{\infty} P_n{}'(x)\, t^n$

$$t\ (1 - 2xt + t^2)^{-\frac{3}{2}} = \sum_{n=0}^{\infty} P_n{}'(x)\, t^n \tag{3}$$

Dividing (2) by (3), we get

$\frac{x-t}{t} = \frac{\sum_{n=0}^{\infty} n\ P_n{}'(x) t^{n-1}}{\sum_{n=0}^{\infty} P_n{}'(x) t^n}$

$(x - t)\sum_{n=0}^{\infty} P_n{}'(x)\, t^n = t.\sum_{n=0}^{\infty} n\ P_n{}'(x)\, t^{n-1} = \sum_{n=0}^{\infty} P_n{}'(x)\, t^n$

Comparing the coefficient of t^n from both sides, we get

$xP_n{}'(x) - P'_{n-1}(x) = n\,P_n(x)$

III. $\boldsymbol{(2n+1)\,P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)}$.

Proof: From relation ***I***, we have

$(n+1)P_{n+1}(x) = (2n+1)P_n(x) - nP_{n-1}(x)$

Differentiating

w.r.t x, we get $5x^3 + x = 2.\frac{1}{2}(5x^3 - 3x) + 4x = 2P_3(x) + 4P_1(x)$

$(n+1)P'_{n+1}(x) =$ $x^3 = \frac{1}{5}[2P_3(x) + 3P_1(x)]$, $x^2 = \frac{1}{3}[2P_2(x) + P_0(x)]$, $x = P_1(x)$, $1 = P_0(x)$

(4) Using

$xP_n{}'(x) - P'_{n-1}($ $x^3 + 2x^2 - x - 3 = \frac{1}{5}[2P_3(x) + 3P_1(x)] + \frac{2}{3}[2P_2(x) + P_0(x)] - P_1(x) - P_0(x)$

Or x $= \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{5}P_1(x) - \frac{7}{3}P_0(x)$

$P_n{}'(x) = n\,P_n(x)$

(5)

Now eliminating the term $xP_n{}'(x)$ from (4) using (5), we get

$(n+1)P'_{n+1}(x) = (2n+1)P_n(x) + (2n+1)[n\,P_n(x) + P'_{n-1}(x)] - nP'_{n-1}(x)$

$(n+1)P'_{n+1}(x) = (n+1)(2n+1)P_n(x) + (n+1)P'_{n-1}(x)$

$P'_{n+1}(x) = (2n+1)P_n(x) + P'_{n-1}(x)$

$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

IV. $\boldsymbol{P_n{}'(x) = x\,P'_{n-1}(x) - n\,P_{n-1}(x)}$.

Proof: Rewriting (4) as

$(n+1)P'_n$

$= (2n+1)P_n(x) + (n+1)x\,P_n{}'(x) + n[x$ $= (n+1)x$ $P'_{n-1}(x) - P'_{n-1}(x)]$

$= (2n+1)P_n(x) + (n+1)x\,P_n{}'(x) + n^2P$ $P'_{n+1}(x) =$ $_n(x)$

$= (n+1)x\,P_n{}'(x) + (n^2 + 2n + 1)P_n(x)$

V. $\boldsymbol{(1-x^2)P_n{}'(x) = n\,[P_{n-1}(x) - x\,P_n(x)]}$.

Proof: From Relation ***II***, we have

$xP_n{}'(x) - P'_{n-1}(x) = n\,P_n(x)$ (6)

Also from relation ***IV***, we have

$P'_n(x) - x\,P_{n-1}{}'(x) = n\,P_{n-1}(x)$ (7)

Multiply equation (7) by x and subtracting form equation (6), we get

$(1-x^2)P_n{}'(x) = n\,[P_{n-1}(x) - x\,P_n(x)]$

ORTHOGONALITY OF LEGENDRE`S POLYNOMIALS

The Legendre Polynomial $P_n(x)$ satisfy the following orthogonality property

$$\int_{-1}^{1} P_m(x).P_n(x)dx = \begin{cases} 0, & m \neq n \\ \dfrac{2}{2n+1}, & m = n \end{cases}$$

Proof: Both of the cases are discussed as follows:

Case I: $\boldsymbol{m \neq n}$

Let the Legendre polynomials $P_m(x)$ and $P_n(x)$ satisfy the differential equations

$(1-x^2)P''_m - 2x\,P'_m + m(m+1)P_m = 0$ (1)

$(1-x^2)P''_n - 2x\,P'_n + n(n+1)P_n = 0$ (2)

Multiplying (1) by $P_n(x)$ and (2) $P_m(x)$ and then subtracting we get

$(1-x^2)[P''_m.P_n - P''_n.P_m] - 2x[P'_m.P_n - P'_n.P_m]$

$+[m(m+1) - n(n+1)]P_m.P_n = 0$

$\frac{d}{dx}[(1-x^2)(P'_m.P_n - P'_n.P_m)] + (m-n)(m+n+1)P_mP_n = 0$

$(m-n)(m+n+1)P_mP_n = -\frac{d}{dx}[(1-x^2)(P'_m.P_n - P'_n.P_m)]$

Integrating from -1 to 1 both sides

$(m-n)(m+n+1)\int_{-1}^{1} P_m(x).P_n(x)dx = -[(1-x^2)(P'_m.P_n - P'_n.P_m)]_{-1}^{1} = 0$

$\int_{-1}^{1} P_m(x).P_n(x)dx = 0$

Case II: $\boldsymbol{m = n}$

We know from generating functions that

$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x)$(3)

Squaring both sides and integrating w.r.t. x from -1 to 1, we get

$\int_{-1}^{1} \frac{1}{1-2xt+t^2}dx = \int_{-1}^{1}[\sum_{n=0}^{\infty} t^n P_n(x)]^2 dx$ (4)

Now $\int_{-1}^{1} \frac{1}{1-2xt+t^2}dx = \left[\frac{\ln(1-2xt+t^2)}{-2t}\right]_{-1}^{1} = -\frac{1}{2t}(\ln(1-2t+t^2) - \ln(1+2t+t^2))$

$= -\frac{1}{2t}(\ln(1-t)^2 - \ln(1+t)^2) = -\frac{1}{t}(\ln(1-t) - \ln(1+t))$

$= \frac{1}{t}(\ln(1+t) - \ln(1-t))$

$= \frac{1}{t}\left[\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots\right) - \left(-t - \frac{t^2}{2} - \frac{t^3}{3} - \cdots\right)\right]$

$= 2\left[1 + \frac{t^2}{3} + \frac{t^4}{5} + \cdots + \frac{t^{2n}}{2n+1} + \cdots\right]$ (5)

Also $\int_{-1}^{1}[\sum_{n=0}^{\infty} t^n P_n(x)]^2 dx = \int_{-1}^{1}[\sum_{n=0}^{\infty} t^n P_n(x)].[\sum_{n=0}^{\infty} t^n P_n(x)]\,dx$

$= \sum_{n=0}^{\infty}\int_{-1}^{1} t^{2n} {P_n}^2(x)\,dx$

$= \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} {P_n}^2(x)\,dx$ (6)

Using (5) and (6) in equation (4), we get

$2\left[1 + \frac{t^2}{3} + \frac{t^4}{5} + \cdots + \frac{t^{2n}}{2n+1} + \cdots\right] = \sum_{n=0}^{\infty} t^{2n} \int_{-1}^{1} {P_n}^2(x)\,dx$

Comparing the coefficient of t^{2n} on both sides we get

$\int_{-1}^{1} {P_n}^2(x)\,dx = \frac{2}{2n+1.}$

FOURIER LEGENDRE EXPANSION

If $f(x)$ be a continuous function and having continuous derivatives over the interval [-1, 1], then we can write

$f(x) = \sum_{n=0}^{\infty} C_n\, P_n(x)$ (1)

To determine the coefficient C_n, multiply both sides by $P_n(x)$ and integrate form -1 to 1, we get

$\int_{-1}^{1} f(x).P_n(x)dx = C_n \int_{-1}^{1} {P_n}^2(x)\,dx$

(Remaining terms vanishes by the orthogonal property)

$= C_n \cdot \frac{2}{2n+1}$

$C_n = \left(n+\frac{1}{2}\right).\int_{-1}^{1} f(x).P_n(x)dx$ (2)

The series in (1) converges uniformly in interval [-1, 1], and is known as Fourier-Legendre Expansion of $f(x)$.

Example 31: Prove that (i) $P'_{2n}(0) = 0$ and $P'_{2n+1}(0) = \frac{(-1)^n(2n+1)}{2^{2n}(n!)^2}$!

Solution: We know $\sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2)^{-\frac{1}{2}}$

Differentiating with respect to $'x'$, we get

$$\sum t^n P'_n(x) = -\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}}(-2t)$$

$$= t(1-2xt+t^2)^{-\frac{3}{2}}$$

Putting $x = 0,\ \sum_{n=0}^{\infty} P'_n(0) = t(1+t^2)^{-\frac{3}{2}}$

$$= t\left\{1-\frac{3}{2}t^2+\frac{-\frac{3}{2}\times-\frac{5}{2}}{2!}t^4+\ \dots\dots+\frac{-\frac{3}{2}\times-\frac{5}{2}\times\dots\dots\left(-\frac{3}{2}-\overline{n-1}\right)}{n!}t^{2n}+\ \dots\right\}$$

Equating the coefficients of t^{2n} and t^{2n+1}, we get $P'_{2n}(0) = 0$

$$P'_{2n+1}(0) = (-1)^n\frac{3\times5\times\dots\dots(2n+1)}{2^n\ n!}$$

$$= (-1)^n\frac{(2n+1)!}{2^n n!^2 2^n}$$

$$P'_{2n+1}(0) = (-1)^n\frac{(2n+1)!}{2^{2n}\ n!^2}$$

Example 32: Prove that

(i) $\big(1$

(ii) $\boldsymbol{(2n+1)(1-x^2)P'_n(x) = n(n+1)[P_{n-}}$

$\boldsymbol{P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)}$

(iii)

Solution: We know $\sum t^n P_n(x) = (1-2xt+t^2)^{-\frac{1}{2}}$

(i) Differentiating with respect to $'t'$ and equating the coefficients of t^n, we will get

$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$ (1)

Now differentiating with respect to $'x'$ and using the derivative with respect $'t'$, we get

$nP_n(x) = xP'_n(x) - P'_{n-1}(x)$ (2)

From (1) & (2), we can derive

$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$ (3)

$P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$ (4)

From (1) & (4) eliminate $P_{n-1}(x)$

$(n+1)P_{n+1}(x) + P'_n(x) = (2n+1)xP_n(x) + xP'_{n-1}(x)$ (5)

$= (2n+1)xP_n(x) + x[xP'_n(x) - nP_n(x)]$, From (4)

$(1-x^2)P'_n(x) = (n+1)xP_n(x) - (n+1)P_{n+1}(x)$
$= (n+1)[xP_n(x) - P_{n+1}(x)]$

(i) Eliminating $P'_{n-1}(x)$ from (2) & (4), we get

$(1-x^2)P'_n(x) = n[P_{n-1}(x) - xP_n(x)]$

$= n\left[P_{n-1}(x) - \frac{1}{2n+1}\{(n+1)P_{n+1}(x) + nP_{n-1}(x)\}\right]$

$= \frac{n}{2n+1}[\{(2n+1)-n\}P_{n-1}(x) - (n+1)P_{n+1}(x)]$

$(2n+1)(1-x^2)P'_n(x) = n(n+1)\{P_{n-1}(x) - P_{n+1}(x)\}$

(ii) (3)−2 ×(2) gives

$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x)$

Example 33: Using the Rodrigue's formula, show that

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}(P_n(x))\right] + n(n+1)P_n(x) = 0$$

Solution: We know that $P_n(x) = \frac{1}{2^n\, n!}\, D^n(x^2-1)^n = \frac{1}{2^n n!}D^nV, \quad V = (x^2-1)^n$

Now differentiating $'V'$ with respect to $'x'$, we get

$V_1 = 2nx(x^2-1)^{n-1}$ or $(x^2-1)V_1 = 2nxV$ or $(1-x^2)V_1 + 2nxV = 0$

Differentiating $(n+1)$ times, we get

$$(1-x^2)V_{n+2} + (n+1)V_{n+1}(-2x) + \frac{(n+1)n}{2!}V_n(-2) + 2nxV_{n+1}$$
$$+(n+1)2nV_n = 0$$

$$(1-x^2)V_{n+2} - 2x\{n+1-n\}V_{n+1} + V_n\{-n(n+1) + 2(n+1)n\} = 0$$

$$(1-x^2)V_{n+2} - 2xV_{n+1} + n(n+1)V_n = 0$$

$$(1-x^2)\frac{d^2}{dx^2}V_n - 2x\frac{d}{dx}V_n + n(n+1)V_n = 0$$

But $V_n = D^nV = 2^n n!\, P_n(x)$

$$(1-x^2)\frac{d^2}{dx^2}[2^n n!\, P_n(x)] - 2x\frac{d}{dx}[2^n n!\, P_n(x)] + n(n+1)2^n n!\, P_n(x) = 0$$

$$(1-x^2)\frac{d^2}{dx^2}P_n(x) - 2x\frac{d}{dx}P_n(x) + n(n+1)P_n(x) = 0$$

Example 34: Prove that

(i) $\int_0^1 P_{2n}(x)dx = 0$

(ii) $\int_{-1}^1 x^m P_n(x)dx = 0 \quad (m < n)$

Solution: (i) we know $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$

$$\therefore \qquad (4n+1)P_{2n}(x) = P'_{2n+1}(x) - P'_{2n-1}(x)$$

Integrating both sides

$$(4n+1)\int_0^1 P_{2n}(x)dx = [P_{2n+1}(x) - P_{2n-1}(x)]_0^1$$

$$= [P_{2n+1}(1) - P_{2n-1}(1)] - [P_{2n+1}(0) - P_{2n-1}(0)]$$

$$= (1-1) - (0-0) = 0$$

(ii) $\int_{-1}^1 x^m P_n(x)dx = \int_{-1}^1 x^m \frac{1}{2^n n!} D^n(x^2-1)^n$

$$= \frac{1}{2^n n!}\left[\{x^m D^{n-1}(x^2-1)^n\}_{-1}^1 - \int_{-1}^1 mx^{m-1}D^{n-1}(x^2-1)^n dx\right]$$

$$= \frac{1}{2^n n!}\left[0 - m\int_{-1}^1 x^{m-1}D^{n-1}(x^2-1)^n dx\right]$$

$$= \frac{1}{2^n n!} \times (-1)^m \int_{-1}^1 D^{n-m}(x^2-1)^n dx$$

$$= \frac{1}{2^n n!}(-1)^m [D^{n-m-1}(x^2-1)^n]_{-1}^1 = 0$$

As $D^{n-m-1}(x-1)^n(x+1)^n = 0$ will contain terms in $(x-1)$ and $(x+1)$ both and hence when $x = \pm$, the value is zero.

Example 35: Prove that $\int_{-1}^1 P_n(x)\left(1 - 2xh + h^2\right)^{-\frac{1}{2}} dx = \frac{2h^n}{2n+1}.$

Solution: We know $(1 - 2xh + h^2)^{-\frac{1}{2}} = \sum h^m P_m(x)$

$$\therefore \qquad \int_{-1}^1 P_n(x)\left(\sum h^m P_m(x)\right)dx = \sum h^m \int_{-1}^1 P_n(x)P_m(x)dx$$

$$= \sum h^m \begin{cases} 0, & n \neq m \\ \frac{2}{2n+1}, & n = m \end{cases} = \frac{2h^n}{2n+1}$$

Example 36: Show that

(i) $\int_{-1}^1 xP_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}$

(ii) $\int_{-1}^1 x^2 P_{n+1}(x)P_{n-1}(x)dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

(iii) $\int_{-1}^1 (1-x^2)[P'_n(x)]^2 dx = \frac{2n(n+1)}{2n+1}$

(iv) $\int_{-1}^1 (1-x^2)P'_m(x)P'_n(x)dx = 0$

Solution: (i) We know $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

$$xP_n(x) = \frac{1}{2n+1}[(n+1)P_{n+1}(x) + nP_{n-1}(x)]$$

$$\therefore \qquad \int_{-1}^1 xP_n(x)P_{n-1}(x)dx$$

$$= \int_{-1}^1 \frac{1}{2n+1}[(n+1)P_{n+1}(x) + nP_{n-1}(x)]P_{n-1}(x)dx$$

$= \frac{n+1}{2n+1}\int_{-1}^{1} P_{n+1}(x)P_{n-1}(x)dx + \frac{n}{2n+1}\int_{-1}^{1} {P_{n-1}}^2(x)\,dx$

$$= \frac{n+1}{2n+1} \times 0 + \frac{n}{2n+1} \times \frac{2}{2n-1} = \frac{2n}{4n^2-1}$$

(ii) We know $xP_n(x) = \frac{1}{2n+1}[(n+1)P_{n+1} + nP_{n-1}(x)]$

Changing $n \to n+1$

$$xP_{n+1}(x) = \frac{1}{2n+1}[(n+2)P_{n+2}(x) + (n+1)P_n(x)]$$

and changing $n \to n-1$

$$xP_{n-1}(x) = \frac{1}{(2n-1)}[nP_n(x) + (n-1)P_{n-2}(x)]$$

$\therefore \quad \int_{-1}^{1} x^2 P_{n+1}(x)P_{n-1}(x)dx$

$= \int_{-1}^{1} \frac{1}{2n+3}[(n+2)P_{n+2}(x) + (n+1)P_n(x)] \times [nP_n(x) + (n-1)P_{n-2}(x)]dx$

$= \frac{1}{(2n-1)(2n+3)}\left[0 + 0 + n(n+1) \times \frac{2}{2n+1} + 0\right]$

$= \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$

(iii) $\int_{-1}^{1}(1-x^2)[P_n'(x)]^2dx = \int_{-1}^{1}(1-x^2)P_n'(x).P_n'(x)$

$= \left[\{(1-x^2)P_n'(x)P_n(x)\}_{-1}^{1} - \int_{-1}^{1}\frac{d}{dx}\{(1-x^2)P_n'(x)\}P_n(x)dx\right]$

$= 0 - \int_{-1}^{1}\{-n(n+1)P_n(x)\}dx = n(n+1)\int_{-1}^{1} P_n^2(x)\,dx = \frac{2n(n+1)}{2n+1}$

(iv) $\int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)dx$

$= \{(1-x^2)P_m'(x)P_n(x)\}_{-1}^{1} - \int_{-1}^{1}\frac{d}{dx}\{(1-x^2)P_m'(x)\}P_n(x)dx$

$$= \left[(0-0) + \int_{-1}^{1} m(m+1)P_m(x)P_n(x)\,dx\right] = m(m+1) \times 0 = 0$$

Example 37: Expand the following functions in terms of Legendre's polynomials in the interval $[1, -1]$

(i) $f(x) = x^3 + 2x^2 - x - 3$ **(ii)** $f(x) = x^4 + x^3 + 2x^2 - x - 3$

Solution: (i) We know $f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$

Where $c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(x)P_n(x)dx$

$\therefore \quad c_0 = \left(0 + \frac{1}{2}\right)\int_{-1}^{1}(x^3 + 2x^2 - x - 3) \times 1\,dx$

$= \frac{1}{2}\left[\left[\frac{x^4}{4} + 2\frac{x^3}{3} - \frac{x^2}{2} - 3x\right]\right]_{-1}^{1} = \frac{1}{2}\left[0 + \frac{4}{3} - 6\right] = \frac{2}{3} - 3 = -\frac{7}{3}$

$c_1 = \left(1 + \frac{1}{2}\right)\int_{-1}^{1}(x^3 + 2x^2 - x - 3)x\,dx = \frac{3}{2}\int_{-1}^{1}(x^4 + 2x^3 - x^2 - 3x)\,dx$

$= -\frac{3}{2}\left(\frac{2}{5} - \frac{2}{3}\right) = 3\left(\frac{3-5}{15}\right) = -\frac{6}{15} = -\frac{2}{5}$

$c_2 = \left(2 + \frac{1}{2}\right)\int_{-1}^{1}(x^3 + 2x^2 - x - 3)\frac{1}{2}(3x^2 - 1)dx$

$= \frac{5}{4}\int_{-1}^{1}[3x^5 - x^3 + 6x^4 - 2x^2 - 3x^3 + x - 9x^2 + 3]dx$

$= \frac{5}{4}\left[6 \times \frac{2}{5} - \frac{4}{3} - \frac{9\times 2}{3} + 6\right] = \frac{4}{3}$

$f(x) = -\frac{7}{3}P_0(x) - \frac{2}{5}P_1(x) + \frac{4}{3}P_2(x) + \ \dots\dots$

(i) $f(x) = x^4 + x^3 + 2x^2 - x - 3 = \sum c_n P_n(x)$

$c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(x)\, p_n(x)dx$

$c_0 = \left(0 + \frac{1}{2}\right)\int_{-1}^{1}(x^4 + x^3 + 2x^2 - x - 3) \times 1\ dx$

$= \frac{1}{2}\left[\frac{2}{5} + \frac{4}{3} - 6\right] = \frac{1}{2} \times \frac{6+20-90}{15} = -\frac{32}{15}$

$c_1 = \left(1 + \frac{1}{2}\right)\int_{-1}^{1}(x^4 + x^3 + 2x^2 - x - 3) \times x\ dx$

$c_1 = \frac{3}{2}\left(\frac{2}{5} - \frac{2}{3}\right) = \frac{3}{2} \times \frac{6-10}{15} = -\frac{2}{5}$

$c_2 = \left(2 + \frac{1}{2}\right)\int_{-1}^{1}(x^4 + x^3 + 2x^2 - x - 3) \times \frac{1}{2}(3x^2 - 1)\ dx$

$= \frac{5}{4}\int_{-1}^{1}[3x^6 - x^4 + 3x^5 - x^3 + 6x^4 - 2x^2 - 3x^3 + x - 9x^2 + 3]dx$

$= \frac{5}{4}\left[\frac{6}{7} - \frac{2}{5} + \frac{12}{5} - \frac{4}{3} - 6 + 6\right] = \frac{40}{21}$

$f(x) = -\frac{32}{15}P_0(x) - \frac{2}{5}P_1(x) + \frac{40}{21}P_2(x)\ +\ \dots\dots$

ASSIGNMENT

1. Show that $P'_n(-x) = (-1)^{n+1}P'_n(x)$.
2. Evaluate the following:

   (i) $\int_0^1 P^2{}_{3n}(x)dx$

   (ii) $\int_{-1}^{1} x^m . P_n(x)dx$.
3. Express $8\ P_5(x) - 8\ P_4(x) - 2\ P_2(x) + 5P_0(x)$ in terms of polynomial of *x*.
4. Use Rodrigues formulae to obtain $P_3(x)$ and $P_4(x)$.
5. Find the value of $\int_0^{\pi/2} \cos t \,.\, P_3(\sin t)\ dt$.
6. Prove that

   (i) $\int_{-1}^{1} \frac{P_n(x)}{\sqrt{1-2xz+z^2}}\ dx = \frac{2z^n}{2n+1}$

   (ii) $\int_{-1}^{1} P_n(x)\ dx = 0$ except when $n = 0$ in which case the value of the integral is 2.

ANSWERS

2.(i) $\frac{1}{6n+1}$

(ii) 0

3. $63\ x^5 - 35\ x^4 - 70x^3 + 27x^2 + 15x + 3$

$P_3(x) = \frac{1}{2}(5x^3 - 3x), P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

4.

5.-1/8

STRUM – LIOUVILLE PROBLEMS

A differential equation of the form

$$[p(x)y']' + [q(x) + \lambda\, r(x)]y = 0 \qquad (1)$$

is called Strum-Liouville Equation where λ is a real number.

Instead of initial conditions, this equation is usually subjected to the boundary conditions on the interval $[a, b]$ as

$$\alpha_1\, y(a) + \alpha_2\, y'(a) = 0,\ \beta_1\, y(b) + \beta_2\, y'(b) = 0 \qquad (2)$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real constants such that either α_1 *or* α_2 are not zero and β_1 *or* β_2 are not zero.

The non trivial solutions of the differential equation (1) subjected to the conditions (2) exists only for specific values of λ, which values are termed as Eigen values or Characteristic values of the equation (1). And the non trivial solution of (1) corresponding to these Eigen values are termed as Eigen functions or Characteristic functions.

ORTHOGONALITY OF EIGEN FUNCTIONS

Two functions $y_m(x)$ and $y_n(x)$ defined on some interval $[a,\ b]$are said to be orthogonal on this interval with respect to the weight function $r(x) > 0$, if

$$\int_a^b r(x)\, y_m(x) y_n(x) dx = 0 \quad for \quad m \neq n$$

Also the norm $\|y_m\|$ of the function $y_m(x)$ is defined to be non negative square root of

$\int_a^b r(x)\,(y_m(x))^2 dx$. Thus $\|y_m\| = \sqrt{\int_a^b r(x)\,(y_m(x))^2 dx}$.

The functions which are orthogonal and having the norm unity are said to be orthonormal functions.

Theorem: ***If $y_m(x)$ and $y_n(x)$ are two eigen functions of the Strum-Liouville problem corresponding to eigen values λ_m and λ_n respectively (where $m \neq n$), then the eigen functions are orthogonal w.r.t. the weight function $r(x)$ over the interval $[a, b]$*.**

Proof: Since distinct eigen values and their corresponding eigen functions are the solutions of the Stum Liouville equation (1), so we can write it as

$$[p(x)y_m']' + [q(x) + \lambda_m\, r(x)]y_m = 0$$

$$[p(x)y_n']' + [q(x) + \lambda_n\, r(x)]y_n = 0$$

Multiplying first equation by y_n and the second equation by y_m, and then subtracting, we get

$$(\lambda_m - \lambda_n)\, r(x) y_m y_n = y_m (r(x) y_n')' - y_n (r(x) y_m')'$$

$$= \frac{d}{dx}\left((r(x)y_n')y_m - (r\,(x)y_m')y_n\right)$$

Now integrating both sides w.r.t.x from a to b, we get

$$(\lambda_m - \lambda_n) \int_a^b r\, y_m y_n\, dx = [(r(x)y_n{}')y_m - (r(x)y_m{}')y_n]_a^b$$

$$= r(b)[y_n{}'(b)y_m(b) - y_m{}'(b)y_n(b)] - r(a)[y_n{}'(a)y_m(a) - y_m{}'(a)y_n(a)]$$

The R.H.S. will vanish if the boundary conditions are of one of the followings forms:

I. $y(a) = y(b) = 0$
II. $y'(a) = y'(b) = 0$
III. $\alpha_1\, y(a) + \alpha_2\, y'(a) = 0,\ \beta_1\, y(b) + \beta_2\, y'(b) = 0$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are real constants such that either α_1 or α_2 are not zero and β_1 or β_2 are not zero. Thus in each of the three cases we get

$$\int_a^b r\, y_m y_n\, dx = 0, \quad (m \neq n)$$

which shows that the eigen functions $y_m(x)$ and $y_n(x)$ are orthogonal w.r.t. the weight function $r(x)$ over the interval $[a, b]$.

Example 38: For Strum-Liouville problem $y'' + \lambda y = 0,\ y(0) = 0,\ y(\pi) = 0$ find the eigen functions.

Solution: For $\lambda = -\gamma^2$, the general solution of the equation is given by

$y(x) = C_1\, e^{\gamma x} + C_2\, e^{-\gamma x}$

Using the above mentioned boundary conditions we get $C_1 = C_2 = 0$. Hence $y(x) = 0$ is not an eigen function.

Also for $\lambda = \gamma^2$, the general solution of the equation is given by

$y(x) = C_1 \cos\gamma x + C_2 \sin\gamma x$

Using $y(0) = 0$, we get $\quad C_1 = 0$

Using $y(\pi) = 0$, we get $\quad C_2 \sin\gamma\pi = 0 \quad => \quad \sin\gamma\pi = 0$

$\therefore \quad \gamma\pi = n\pi \quad => \quad \gamma = n,\ n = \pm 1,\ \pm 2,\ \pm 3, ...$

Thus the eigen values are $\lambda = 0, 1, 4, 9, ...$ and taking $C_2 = 1$, we obtain the eigen functions as

$y_n(x) = \sin nx,\ n = 0, 1, 2, ...$

ASSIGNMENT

Find the eigen values of each of the following Stum Liouville problems and prove their orthogonality:

i) $y'' + \lambda y = 0,\ y(0) = 0,\ y(l) = 0$

ii) $y'' + \lambda y = 0,\ y'(0) = 0,\ y'(c) = 0$

iii) $y'' + \lambda y = 0,\ y(\pi) = y(-\pi),\ y'(\pi) = y'(-\pi)$

1. Show that the eigen values of the boundary value problem $y'' + \lambda y = 0,\ y(0) = 0,\ y(\pi) + y'(\pi) = 0$ satisfies $\sqrt{\lambda} + \tan\sqrt{\lambda}\,\pi = 0$.

ANSWERS

1.(i) $\sin\frac{n\pi x}{l}, \; n = 0, 1, 2, \ldots$

(ii) $\cos\frac{n\pi x}{c}, \; n = 0, 1, 2, \ldots$

(iii) $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$

UNIT – IV

PARTIAL DIFFERENTIAL EQUATIONS

Definition:- PDE

A *partial differential equation (or briefly a PDE)* is a mathematical equation that involves *two or more independent variables* and an *unknown function* of two or more variables (depend on those variables) and *partial derivatives* of the unknown function with respect to the *independent variables.*

Applications:-

Partial differential equations are used to mathematically formulate, and thus aid the solution of, physical and other problems involving functions of several variables, such as the *propagation of heat or sound, fluid flow, elasticity, electrostatics, electrodynamics*, etc.

Notations:

We use the following notations to denote partial derivatives

$$p = \frac{\partial z}{\partial x} = z_x;\ q = \frac{\partial z}{\partial y} = z_y;\ r = \frac{\partial^2 z}{\partial x^2} = z_{xx};\ s = \frac{\partial^2 z}{\partial x\, \partial y} = z_{xy};\ t = \frac{\partial^2 z}{\partial y^2} = z_{yy}$$

Examples:-

1. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$, (u – dependent variable; x, y – independent variables)

2. $\left(\frac{\partial u}{\partial x}\right)^3 + \frac{\partial u}{\partial y} = 0$, (u – dependent variable; x, y – independent variables)

3. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = 0$, (u – dependent variable; x, y and t – independent variables)

Definition:- ORDER OF PDE

derivative involved in the given PDE. The *order* of a *partial differential equation* is the order of the *highest*

Example:-

1. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$, is a second order equation in two variables.

2. $\left(\frac{\partial u}{\partial x}\right)^3 + \frac{\partial u}{\partial y} = 0$, is a first order equation in two variables

3. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = 0$, is a first order equation in three variables.

Definition:- PARTICULAR SOLUTION

A *solution (or a particular solution)* to a partial differential equation is a function that solves the equation.

Definition:- GENERAL SOLUTION

A solution is called *general* if it contains all particular solutions of the PDE equation concerned.

LINEAR PARTIAL DIFFERENTIAL EQUATION

If the dependent variable and its partial derivatives occur in the first degree, then we say that the partial differential equation is linear.

Examples:-

1. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$, (Linear PDE)

2. $\left(\frac{\partial u}{\partial x}\right)^3 + \frac{\partial u}{\partial y} = 0$, (NON-Linear PDE, since degree is three u_x)

3. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial t} = 0$, (Linear PDE)

Formation of Partial Differential Equations

Partial differential equations can be obtained by the elimination of *arbitrary constants* or by the elimination of *arbitrary functions*.

By the Elimination of Arbitrary Constants

Let us consider the function

$$\varphi(x, y, z, a, b) = 0 \qquad (1)$$

where a & b are arbitrary constants

Differentiating equation (1) partially w.r.t *x* & *y*, we get

$$\frac{\partial \varphi}{\partial x} + p\frac{\partial \varphi}{\partial z} = 0 \qquad (2)$$

$$\frac{\partial \varphi}{\partial y} + q\frac{\partial \varphi}{\partial z} = 0 \qquad (3)$$

Eliminating *a* and *b* from equations (1), (2) and (3), we get a partial differential equation of the first order of the form $f(x, y, z, p, q) = 0$

Problem:-0 1

Eliminate the arbitrary constants *a* & *b* from $z = ax + by + ab$

Solution:-

Consider $z = ax + by + ab \qquad (1)$

Differentiating (1) partially w.r.t *x* & *y*, we get

$$\frac{\partial z}{\partial x} = a \qquad \text{i.e.,} \quad p = a \ (2)$$

$$\frac{\partial z}{\partial y} = b \qquad \text{i.e.,} \quad q = b \qquad (3)$$

Using (2) & (3) in (1), we get $z = px + qy + pq$

which is the required partial differential equation.

Problem:-0 2

Form the partial differential equation by eliminating the arbitrary constants *a* and *b* from $z = (x^2 + a^2)(y^2 + b^2)$

Solution:-

Given $z = (x^2 + a^2)(y^2 + b^2)$ (1)

Differentiating (1) partially w.r.t *x* & *y*, we get

$$p = 2x(y^2 + b^2)$$

$\Rightarrow p/2x = (y^2 + b^2)$-----(2)

$$q = 2y(x^2 + a^2)$$

$\Rightarrow q/2y = (x^2 + a^2)$-----(3)

Substituting the values of p and q in (1), we get $4xyz = pq$

which is the required partial differential equation.

Problem:-0 3

Find the partial differential equation of the family of spheres of radius one whose center lie in the *xy* - plane.

Solution:-

The equation of the sphere is given by

$(x - a)^2 + (y - b)^2 + z^2 = 1$ (1)

Differentiating (1) partially w.r.t *x* & *y*, we get

$$\frac{\partial z^n}{\partial x} = \frac{\partial z^n}{\partial z}\frac{\partial z}{\partial x} = nz^{n-1}p$$

$2(x - a) + 2zp = 0$------(2)

$$\frac{\partial z^n}{\partial y} = \frac{\partial z^n}{\partial z}\frac{\partial z}{\partial y} = nz^{n-1}q$$

$2(y - b) + 2zq = 0$----(3)

From these equations we obtain

$$x - a = -zp \qquad (2)$$

$$y - b = -zq \qquad (3)$$

Using (2) and (3) in (1), we get

$$z^2p^2 + z^2q^2 + z^2 = 1 \quad \text{(or)} \quad z^2(p^2 + q^2 + 1) = 1$$

Problem:-04

Eliminate the arbitrary constants *a, b & c* from $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$and form the partial differential equation.

Solution:-

The given equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \qquad (1)$

Differentiating (1) partially w.r.t *x* & *y*, we get

$$\frac{2x}{a^2} + \frac{2zp}{c^2} = 0$$

$$\frac{2y}{b^2} + \frac{2zq}{c^2} = 0$$

Therefore, we get

$$\frac{x}{a^2} + \frac{zp}{c^2} = 0 \qquad (2)$$

$$\frac{y}{b^2} + \frac{zq}{c^2} = 0 \qquad (3)$$

Again differentiating (2) partially w.r.t *x*, we get

$$\frac{1}{a^2} + \left(\frac{1}{c^2}\right)(zr + p^2) = 0 \qquad (4)$$

Multiplying (4) by *x*, we get

$$\frac{x}{a^2} + \frac{xzr}{c^2} + \frac{p^2x}{c^2} = 0$$

From (2), we have

$$\frac{-zp}{c^2} + \frac{xzr}{c^2} + \frac{p^2x}{c^2} = 0$$

$$\text{(or)} -zp + xzr + p^2x = 0$$

By the Elimination of Arbitrary Functions

Let u and v be any two functions of x, y, z and $\varphi(u, v) = 0$, where φ is an arbitrary function. This relation can be expressed as

$$u = f(v) \qquad (1)$$

Differentiating (1) partially w.r.t *x* & *y* and eliminating the arbitrary functions from these relations, we get a partial differential equation of the first order of the form $f(x, y, z, p, q) = 0$.

Problem:-05

Obtain the partial differential equation by eliminating *f* from

$z = (x + y) f(x^2 - y^2)$

Solution:-

Let us now consider the equation

$$z = (x + y) f(x^2 - y^2) \qquad (1)$$

Differentiating (1) partially w.r.t *x* & *y*, we get

$p = (x + y) f'(x^2 - y^2) . 2x + f(x^2 - y^2)$

$q = (x + y) f'(x^2 - y^2) . (-2y) + f(x^2 - y^2)$

These equations can be written as

$$p - f(x^2 - y^2) = (x + y) f'(x^2 - y^2).2x \qquad (2)$$

$$q - f(x^2 - y^2) = (x + y) f'(x^2 - y^2). (-2y) \qquad (3)$$

Hence, we get $\frac{p-f(x^2-y^2)}{q-f(x^2-y^2)} = -\frac{x}{y}$

$$\text{ie., } py - yf(x^2 - y^2) = -qx + xf(x^2 - y^2)$$

ie., $py + qx = (x + y) f(x^2 - y^2)$

Therefore, we have by(1), $py + qx = z$

Problem:- 06

Form the partial differential equation by eliminating the arbitrary function *f* from $z = e^y f(x + y)$

Solution:-

Consider $z = e^y f(x+y)$ (1)

Differentiating (1) partially w.r.t x & y, we get

$p = e^y f'(x+y)(1+0) = e^y f'(x+y)$-----(2)

$q = e^y f'(x + y)(0+1) + f(x + y).\, e^y$ ----(3)

Hence, we have

(3)-(2)=>$q - p = z$

$q+p=z$

Problem:- 07

Form the PDE by eliminating *f*&φ from $z = f(x + ay) + \varphi(x - ay)$

Solution:-

Consider $z = f(x + ay) + \varphi(x - ay)$ (1)

Differentiating (1) partially w.r.t *x* & *y* , we get

$p = f'(x+ay)(1+0) + \varphi'(x-ay)(1-0) = f'(x+ay) + \varphi'(x-ay)$ (2)

$q = f'(x+ay)(0+a) + \varphi'(x-ay)(0-a)$ (3)

Differentiating (2) & (3) again partially w.r.t *x* & *y*, we get

$r = f''(x+ay)(1+0) + \varphi''(x-ay)(1+0) = f''(x+ay) + \varphi''(x-ay)$----(4)

$t = f''(x+ay)(0+a)a + \varphi''(x-ay)(0-a)(-a) = f''(x+ay)\, a^2 + \varphi''(x-ay)a^2$---(5)

Sub (4) in (5)

i.e., $t = a^2\{f''(x + ay) + \varphi''(x - y)\}$

(or) $t = a^2 r$

EXERCISES

1. Form the partial differential equation by eliminating the arbitrary constants *a* & *b* from the following equations.

(i) $z = ax + by$

(ii) $\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$

(iii) $z = ax + by + \sqrt{a^2 + b^2}$

(iv) $ax^2 + by^2 + cz^2 = 1$

(v) $z = a^2x + b^2y + ab$

2. Find the PDE of the family of spheres of radius 1 having their centers lie on the *xy* plane {Hint: $(x - a)^2 + (y - b)^2 + z^2 = 1$}

3. Find the PDE of all spheres whose center lie on the (i) *z* axis (ii)*x*-axis

Form the partial differential equations by eliminating the arbitrary functions in the following cases.

(i) $z = f(x + y)$

(ii) $z = f(x^2 - y^2)$

(iii) $z = f(x^2 + y^2 + z^2)$

(i v) $f(xyz, x + y + z) = 0$

(v) $z = x + y + f(xy)$

(vi) $z = xy + f(x^2 + y^2)$

(v) $z = f\left(\frac{xy}{z}\right)$

(vi) $f(xy + z^2, x + y + z) = 0$

(vii) $z = f(x + iy) + f(x - iy)$ (x)

(xi) $z = f(x^3 + 2y) + g(x^3 - 2y)$

TYPES OF SOLUTIONS OF PDE

Complete Integral

A solution containing as many arbitrary constants as there are independent variables is called a complete integral.

i.e if the partial differential equations contain only two independent variables so that the complete integral will have two constants.

Particular Integral

A solution obtained by giving particular values to the arbitrary constants is called a particular integral.

Singular Integral

Let $f(x,y,z,p,q) = 0$ (1)

be the partial differential equation whose complete integral is

$\varphi(x,y,z,a,b) = 0$ (2)

where *a* and *b* are arbitrary constants.

Differentiating (2) partially w.r.t. *a* and *b*, we obtain

$\frac{\partial \varphi}{\partial a} = 0$ (3)

and $\frac{\partial \varphi}{\partial b} = 0$ (4)

The elimination of *a* and *b* from the equations (2), (3) and (4), when it exists, is called the singular integral of (1).

General Integral

In the complete integral (2), put $b = F(a)$, we get

φ(x,y,z,a,F(a))=0 (5)

Differentiating (2), partially w.r.t. *a*, we get

$\frac{\partial\varphi}{\partial a} + \frac{\partial\varphi}{\partial b}F'(a) = 0$ (6)

The eliminate of *a* between (5) and (6), if it exists, is called the general integral of (1).

SOLUTION OF FIRST ORDER PDE

The first order partial differential equation can be written as $f(x, y, z, p, q) = 0$, where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$.

TYPE - I

f (p,q) = 0. i.e., equations containing p and q only.

Suppose that z = ax + by +c is a solution of the equation f(p,q) = 0, where f (a,b)= 0.

Solving this for b, we get b = F (a).

Hence the complete integral is z = ax + F(a)y+ c (1)

Now, the singular integral is obtained by eliminating a & c between

z = ax + y F(a) + c 0 = x + y F'(a)

0 = 1.

The last equation being absurd, the singular integral does not exist in this case.

To obtain the general integral, let us take c = φ(a).

Then, z = ax + F(a) y +φ(a)(2)

Differentiating (2) partially w.r.t. a, we get

0 = x + F'(a). y+φ'(a)(3)

The eliminate of *a* between (2) and (3), we get the general integral

Problem:-0 8

Solve $pq = 2$

Solution:-

The given PDE is $pq - 2 = 0$

The given equation is of the form $f(p,q) = 0$

The solution is $z = ax + by + c$, where $ab = 2$.

To find complete integral

Solving, $b = \frac{2}{a}$

The complete integral is $z = ax + \frac{2}{a}y + c$ (1)

To find singular integral

Differentiating (1) partially w.r.t *c*, we get 0 = 1,which is absurd.

Hence, there is *no singular integral*.

To find the general integral,

put $c = \varphi(a)$ in (1), we get $z = ax + \frac{2}{a}y + \varphi(a)$

Differentiating partially w.r.t *a*, we get

$$0 = x - \frac{2}{a^2}y + \varphi'(a)$$

Eliminating *a* between these equations gives the general integral.

Problem :-09

Solve $pq + p + q = 0$

Solution:-

The given equation is of the form $f(p, q) = 0$.

The solution is $z = ax + by + c$, where $ab + a + b = 0$.

To find complete integral

Solving, we get $b = -\frac{a}{1+a}$

Hence the complete Integral is $z = ax - \frac{a}{1+a}y + c$ (1)

To find the singular integral

Differentiating (1) partially w.r.t. *c*, we get 0 = 1.

The above equation being absurd, there is no singular integral for the given partial differential equation.

To find the general integral

Put $c = \varphi(a)$ in (1), we have

$$z = ax - \frac{a}{1+a}y + \varphi(a) \qquad (2)$$

Differentiating (2) partially w.r.t *a*, we get

$$0 = x - [\frac{a(0+1)-(1+a).1}{(1+a)^2}] + \varphi'(a)$$

$$0 = x - \frac{-1}{(1+a)^2} + \varphi'(a) \qquad (3)$$

Eliminating *a* between (2) and (3) gives the general integral.

Problem:-10

Solve$p^2 + q^2 = npq$

Solution:-

The given PDE is $p^2 + q^2 - npq=0$ ---(1)

It is of the form f(p,q)=0

The solution of this equation is z = ax + by + c, where $a^2 + b^2 = nab$.

To find complete integral

$b^2 - nab + a^2 = 0$

Solving, we get

$$b = a\left(\frac{n \pm \sqrt{n^2 - 4}}{2}\right)$$

Hence the complete integral is $z = ax + a\left(\frac{n\pm\sqrt{n^2-4}}{2}\right)y + c$ (1)

To find singular integral

Differentiating (1) partially w.r.t *c*, we get 0 = 1, which is absurd.

Therefore, there is no singular integral for the given equation.

To find the general Integral,

Put $c = \varphi(a)$, we get

$$z = ax + a\left(\frac{n \pm \sqrt{n^2-4}}{2}\right)y + \varphi(a),$$

Differentiating partially w.r.t *a*, we have $0 = x + \left(\frac{n \pm \sqrt{n^2-4}}{2}\right)y + \varphi'(a)$

The eliminate of *a* between these equations gives the general integral.

TYPE - II

Equations of the form $f(x, p, q) = 0$, Or $f(y, p, q) = 0$ and $f(z, p, q) = 0$.

i.e, one of the variables x, y, z occurs explicitly.

(i) Let us consider the equation $\mathbf{f(x, p, q) = 0}$.

Since *z* is a function of *x* and *y*, we have

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$dz = pdx + qdy$----(1) gives the solution of given equation.

Substitute $q = a$, the given equation takes the form $f(x, p, a) = 0$

Solving, we get $p = \varphi(x, a)$.

Substitute the value of p,q in (1), we get

$dz = \varphi(x, a)dx + ady$.

Integrating,

$z = \int \varphi(x, a)dx + ay + b$

which is a complete Integral.

The singular and general integrals are found in the usual way.

(ii) Let us consider the equation $f(y, p, q) = 0$.

Substitute $p = a$ in given equation, the equation becomes $f(y, a, q) = 0$

Solving, we get $q = \varphi(y, a)$.

Substitute the value of p,q in (1), we get

$dz = adx + \varphi(y, a)dy$.

Integrating on both sides,

$z = ax + \int \varphi\,(y, a)\,dy + b$,

which is a complete Integral.

The singular and general integrals are found in the usual way.

(iii) Let us consider the equation $f(z, p, q) = 0$.

Substitute $q = ap$. in given equation, the equation becomes $f(z, p, ap) = 0$

Solving, we get $p = \varphi(z, a)$.

Substitute p,q in (1) , we get

$dz = \varphi(z, a)dx + a\,\varphi(z, a)dy$

$$\text{i.e.,}\ \frac{dz}{\varphi(z, a)} = dx + ady$$

Integrating,

$$\int \frac{dz}{\varphi(z,a)} = x + ay + b \quad .$$

Which is complete integral.

The singular and general integrals are found in the usual way.

Problem:- 11

Find the complete integral of $q = xp + p^2$

Solution:-

Given $q = xp + p^2$ -----(1)

This is of the form $f(x, p, q) = 0$.

The complete solution is given by dz = pdx+qdy-----(2)

Put q = a in (1), we get

(1) =>a = xp + p^2

i.e., $p^2 + xp - a = 0$

$$p = \frac{-x \pm \sqrt{(x^2+4a)}}{2}$$

Substitute p, q in (2), we get

$$dz = \frac{-x \pm \sqrt{(x^2+4a)}}{2}dx + ady$$

Integrating on both sides, we get

$$\int dz = \int\left(-\frac{x}{2} \pm \frac{\sqrt{(x^2+2^2(\sqrt{a})^2)}}{2}\right)dx + a\int dy + C$$

We know that $\int\sqrt{x^2+b^2}dx = \frac{x}{2}\sqrt{x^2+b^2} + \frac{a^2}{2}In\left|x+\sqrt{x^2+b^2}\right|$

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left(\frac{x}{2}\sqrt{4a+x^2} + \frac{4a}{2}\ \text{In}\ |x + \sqrt{4a+x^2}|\right) + ay + C.$$

Which is the complete integral.

Note

Singular Integral

Differentiate complete integral p.w.r.t C, we get 0=1, there is no singular integral.

General Solution

Let C=$\phi(a)$, substitute in complete integral

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left(\frac{x}{2}\sqrt{4a+x^2} + \frac{4a}{2}\ \text{In}\ |x + \sqrt{4a+x^2}|\right) + ay + \emptyset(a).$$

Differentiate the above p.w.r.t 'a', and eliminate 'a' between the equation, we get the general integral.

Problem:- 12

Solve $q = p^2 y$

Solution:-

Given $q=p^2y$----(1)

This is of the form $f(y, p, q) = 0$

The complete solution is given by $dz = pdx + qdy$----(2)

put $p = a$. in (1)

Therefore, the given equation becomes $q = a^2 y$.

Substitute p,q in (2), we get

$dz = adx + a^2 y\, dy$

Integrating on both sides, we get

$$\int dz = a\int dx + a^2 \int ydy + C$$

$z=ax+a^2y^2/2+C$

Which is the complete solution.

Note

Singular Integral

Differentiate complete integral p.w.r.t C, we get 0=1, there is no singular integral.

General Solution

Let $C=\phi(a)$, substitute in complete integral

$z=ax+a^2y^2/2+\emptyset(a)$.

Differentiate the above p.w.r.t a,

$0=x+ay^2+\emptyset'(a)$

Eliminate 'a' between the equation, we get the general integral.

Problem:- 13

Solve $9(p^2z + q^2) = 4$

Solution:-

Given $9(p^2z + q^2) = 4$----(1)

This is of the form $f(z, p, q) = 0$

The complete solution is given by dz=pdx+qdy----(2)

put $q = ap$, the given equation becomes

$9(p^2z + a^2p^2) = 4$

$p^2(z+a^2)=4/9$

$p^2=4/9(z+a^2)$

Therefore, $p = \pm \frac{2}{3\sqrt{z+a^2}}$

and $q = \pm \frac{2a}{3\sqrt{z+a^2}}$

Substitute p,q in (2), we get

$$dz = \pm \frac{2}{3\sqrt{z + a^2}} dx \pm \frac{2a}{3\sqrt{z + a^2}} dy$$

Multiplying both sides by $\sqrt{z + a^2}$, we get

$\sqrt{z + a^2} dz = \pm \frac{2}{3} dx \pm \frac{2}{3} a\, dy,$

Integrate on both sides, we get

We know that $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$

$\frac{(z + a^2)^{3/2}}{3/2} = \pm \frac{2}{3} x \pm \frac{2}{3} ay + C$

$(z + a^2)^{3/2} = x + ay + 3C/2.$

Note

Singular Integral

Differentiate complete integral p.w.r.t C, we get 0=3/2, there is no singular integral.

General Solution

Let C=$\phi(a)$, substitute in complete integral

$(z + a^2)^{3/2} = x + ay + (3/2)\emptyset(a)$.

Differentiate the above p.w.r.t a, and eliminate 'a' between the equation, we get the general integral.

TYPE - III

$f_1(x, p) = f_2(y, q)$.

i.e., equations in which 'z' is absent and the variables are separable.

Let us assume as a trivial solution that

$f(x, p) = g(y, q) = a$ (say).

Solving for *p* and *q*, we get

$p = F(x, a)$ and $q = G(y, a)$.

But $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

Hence $dz = pdx + qdy$

$dz = F(x, a)dx + G(y, a)\, dy$

Therefore, $z = \int F(x,a)\, dx + \int G(y,a)\, dy + b$,

which is the complete integral of the given equation containing two constants a and b.

The singular and general integrals are found in the usual way.

Problem:-14

Find the complete integral of $pq = xy$

Solution:-

Given pq=xy

The given equation can be written as

$$\frac{p}{x} = \frac{y}{q}$$

It is of the form f(x,p)=g(y,q)

The complete solution is given by dz=pdx+qdy----(2)

$\frac{p}{x} = \frac{y}{q} = a$ (say)

Therefore, $\frac{p}{x} = a \Rightarrow p = ax$

and $\frac{y}{q} = a \Rightarrow q = \frac{y}{a}$

Substitute p,q value in (2), we get

$dz = ax\,dx + \frac{y}{a}dy$,

which on integration gives

$$z = \frac{ax^2}{2} + \frac{y^2}{2a} + C$$

Problem:- 15

Find the complete integral of $p^2 + q^2 = x^2 + y^2$

Solution:-

Given $p^2 + q^2 = x^2 + y^2$----(1)

The given equation can be written as $p^2 – x^2 = y^2 – q^2$

It is of the form f(x,p)=g(y,q)

The complete solution is given by dz=pdx+qdy----(2)

$p^2 – x^2 = y^2 – q^2 = a^2$ (say)

$p^2 - x^2 = a^2$ implies $p = \pm\sqrt{a^2 + x^2}$

$y^2 - q^2 = a^2$ implies $q = \pm\sqrt{y^2 - a^2}$

Substitute p,q in (2), we get

$$dz = \pm\sqrt{a^2 + x^2}dx \pm \sqrt{y^2 - a^2}dy$$

Integrating, we get

$$z = \pm\left(\frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sin h^{-1}\left(\frac{x}{a}\right)\right) \pm \left(\frac{y}{2}\sqrt{y^2 - a^2} - \frac{a^2}{2}\cos h^{-1}\left(\frac{y}{a}\right)\right) + C$$

TYPE - IV (Clairaut's form)

Equation of the type $z = px + qy + f(p, q) \to (1)$ is known as Clairaut's form.

Differentiating (1) partially w.r.t x and y, we get

$p = a$ and $q = b$.

Therefore, the complete integral is given by

$z = ax + by + f(a, b)$.

Problem:- 16

Solve $z = px + qy + pq$

Solution:-

Given $z = px + qy + pq$----(1)

The given equation is in Clairaut's form z=px+qy+f(p,q)

Putting p = a and q = b in (1), we have

$z = ax + by + ab$ (2)

which is the complete integral.

To find the singular integral

Differentiating (2) partially w.r.t *a* and *b*, we get

$0 = x + b;$

$0 = y + a$

Therefore, we have

$a = -y$

and $b = -x$.

Substituting the values of *a* & *b* in (2), we get

$$z = -xy - xy + xy$$

or $\quad z + xy = 0,$

which is the singular integral.

To get the general integral,

put $b = \varphi(a)$ in (1).

Then $z = ax + \varphi(a)y + a\,\varphi(a) \qquad (3)$

Differentiating (3) partially w.r.t *a*, we have

$0 = x + \varphi'(a)y + a\,\varphi'(a) + \varphi(a) \qquad (4)$

Eliminating *a* between (3) and (4), we get the general integral.

Problem:- 17

Find the complete and singular solutions of $z = px + qy + \sqrt{1 + p^2 + q^2}$

Solution:-

Given $z = px + qy + \sqrt{1 + p^2 + q^2}$---(1)

It is in Clairaut's form z=px+qy+f(p,q)

Sub p=a and q=b in (1), the complete integral is given by

$z = ax + by + \sqrt{1 + a^2 + b^2} \qquad (2)$

To obtain the singular integral,

Differentiating (1) partially w.r.t *a* & *b*.

Then, $0 = x + \frac{a}{\sqrt{1+a^2+b^2}}$ and $0 = y + \frac{b}{\sqrt{1+a^2+b^2}}$

Therefore, $x = \frac{-a}{\sqrt{1+a^2+b^2}}$ (3)

$y = \frac{-b}{\sqrt{1+a^2+b^2}}$ (4)

Squaring (3) & (4) and adding, we get

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2} = \frac{1+a^2+b^2-1}{1+a^2+b^2}$$

$$x^2 + y^2 = 1 - \frac{1}{1 + a^2 + b^2}$$

$$\frac{1}{1 + a^2 + b^2} = 1 - (x^2 + y^2)$$

Now, $1 - x^2 - y^2 = \frac{1}{1+a^2+b^2}$

i.e., $1 + a^2 + b^2 = \frac{1}{1-x^2-y^2}$

Therefore, $\sqrt{1 + a^2 + b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$ (4)

Using (4) in (2) & (3), we get

$x = -a\sqrt{1 - x^2 - y^2}$ and $y = -b\sqrt{1 - x^2 - y^2}$

Hence, $a = \frac{-x}{\sqrt{1-x^2-y^2}}$ and $b = \frac{-y}{\sqrt{1-x^2-y^2}}$

Substituting the values of *a&b* in (1), we get

$$z = \frac{-x^2}{\sqrt{1 - x^2 - y^2}} - \frac{y^2}{\sqrt{1 - x^2 - y^2}} + \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$z = \frac{1 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}}$$

$$z = \sqrt{1 - x^2 - y^2}$$

$x^2 + y^2 + z^2 = 1$, which is the singular integral.

EXERCISES

Solve the following Equations

1. $pq = k$
2. $p + q = pq$
3. $\sqrt{p} + \sqrt{q} = x$
4. $p = y^2q^2$
5. $z = p^2 + q^2$
6. $p + q = x + y$
7. $p^2z^2 + q^2 = 1$
8. $z = px + qy - 2\sqrt{pq}$
9. $\{z - (px + qy)\}^2 = c^2 + p^2 + q^2$
10. $z = px + qy + p^2q^2$

EQUATIONS REDUCIBLE TO THE STANDARD FORMS

Sometimes, it is possible to have non – linear partial differential equations of the first order which do not belong to any of the four standard forms discussed earlier. By using the suitable substitution, we can reduce them into linear PDE and in any one of the four types, then it can be solved using usual procedure and by back substitution, obtain the solution of given non linear PDE.

Type (i):

Equations of the form $F(x^m p, y^n q) = 0$ or $F(z, x^m p, y^n q) = 0$.

Case(i):

If $m \neq 1$ and $n \neq 1$, then put $x^{1-m} = X$ and $y^{1-m} = Y$.

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}(1-m)x^{-m}$

Therefore, $x^m p = \frac{\partial z}{\partial X}(1-m) = (1-m)P$, where $P = \frac{\partial z}{\partial X}$

Similarly, $y^n q = (1-n)Q$, where $Q = \frac{\partial z}{\partial Y}$

Hence, the given equation takes the form $F(P,Q) = 0$ (or) $F(z,P,Q) = 0$.

Case(ii):

If m = 1 and n = 1,then put log x = X and log y = Y.

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = \frac{\partial z}{\partial X}\frac{1}{x}$

Therefore, $xp = \frac{\partial z}{\partial X} = P$

Similarly, $yq = Q$.

Problem:-18

Find the complete solution of $x^4p^2 + y^2zq = 2z^2$

Solution:-

Given $x^4p^2 + y^2zq = 2z^2$

The given equation can be expressed as $(x^2p)^2 + (y^2q)z = 2z^2$----(1)

It is of the form $f(z, x^m p, y^n q)=0$, where m = 2, n = 2

Put $X = x^{1-m} = x^{-1}$ and $Y = y^{1-n} = y^{-1}$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P\frac{\partial(x^{-1})}{\partial x} = -x^{-2}P$$

$$x^2 p = -P$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = P\frac{\partial(y^{-1})}{\partial y} = -y^{-2}Q$$

$$y^2 q = -Q$$

Substitute the above in (1), we have

$P^2 - Qz = 2z^2$ (2)

This equation is of the form $f(z,P,Q) = 0$.

The complete solution is given by dz=PdX+QdY-----(3)

Let $Q = aP$ in (2), we have

$P^2 - aPz = 2z^2$

$P^2 - azP - 2z^2 = 0$

$$P = \frac{-(-az) \pm \sqrt{(-za)^2 - 4(1)(-2z^2)}}{2(1)}$$

$$P = \frac{az \pm \sqrt{(za)^2 + 8z^2)}}{2}$$

$$P = \frac{az \pm z\sqrt{a^2 + 8}}{2}$$

Hence $Q = a\left(\frac{a \pm \sqrt{a^2+8}}{2}\right)z$

Substitute P, Q in (3), we have

$$dz = \left(\frac{a \pm \sqrt{a^2 + 8}}{2}\right)zdX + a\left(\frac{a \pm \sqrt{a^2 + 8}}{2}\right)zdY$$

Divide the above equation by z, we have

i.e., $\frac{dz}{z} = \left(\frac{a \pm \sqrt{a^2+8}}{2}\right)(dX + adY)$

Integrate on both sides

i.e., $\int \frac{dz}{z} = \left(\frac{a \pm \sqrt{a^2+8}}{2}\right)\int(dX + adY)$

$$\log z = \left(\frac{a \pm \sqrt{a^2 + 8}}{2}\right)(X + aY) + C$$

Substitute $X = x^{-1}$ and $Y = y^{-1}$

$$\log z = \left(\frac{a \pm \sqrt{a^2 + 8}}{2}\right)\left(\frac{1}{x} + \frac{a}{y}\right) + C$$

which is the complete solution.

Problem- 19

Find the complete solution of $x^2p^2 + y^2q^2 = z^2$

Solution:-

Given $x^2p^2 + y^2q^2 = z^2$

The given equation can be written as $(xp)^2 + (yq)^2 = z^2$ ---(1)

This equation is of the form $f(z, x^m p, y^n q) = 0$. where m = 1, n = 1.

Put X = log x and Y = log y.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} = P\frac{\partial (\log x)}{\partial x} = \frac{1}{x}P$$

=>xp = P.

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = Q\frac{\partial (\log y)}{\partial y} = \frac{1}{y}Q$$

=>yq = Q

Hence the given equation becomes

$P^2 + Q^2 = z^2$ (2)

This equation is of the form $F(z, P, Q) = 0$.

The complete solution is given by dz=PdX+QdY-----(3)

Put Q = aP. in equation (2) becomes,

$P^2 + a^2 P^2 = z^2$

$(1 + a^2)P^2 = z^2$

$$P = \frac{z}{\sqrt{1+a^2}}$$

and $$Q = \frac{az}{\sqrt{1+a^2}}$$

Sub p,q in (3),we get

$$dz = \frac{z}{\sqrt{1 + a^2}}dX + \frac{az}{\sqrt{1 + a^2}}dY$$

i.e., $\left(\sqrt{1 + a^2}\right)\frac{dz}{z} = dX + adY$.

Integrate on both sides

i.e.,$\left(\sqrt{1+a^2}\right)\int \frac{dz}{z} = \int dX + a\int dY$.

$\left(\sqrt{1+a^2}\right)\log z = X + aY + C$

Sub X=log x and Y=log y

$\left(\sqrt{1+a^2}\right)\log z = \log x + a\log y + C,$

which is the complete solution.

Type (ii):

Equations of the form $F(z^k p, z^k q) = 0$ (or) $F(x, z^k p) = G(y, z^k q)$.

Case (i):

If $k \neq 1$, put $Z = z^{k+1}$,

Now $\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}.\frac{\partial z}{\partial x} = (k+1)z^k.\frac{\partial z}{\partial x} = (k+1)z^k p$

Therefore, $z^k p = \frac{1}{k+1}.\frac{\partial Z}{\partial x}$

Similarly, $z^k q = \frac{1}{k+1}.\frac{\partial Z}{\partial y}$

Case (ii):

If k = -1, put Z = log z.

Now, $\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}.\frac{\partial z}{\partial x} = \frac{1}{z}p$

Similarly, $\frac{\partial Z}{\partial y} = \frac{1}{z}q.$

Problem:-20

Solve $z^4q^2 - z^2p = 1$

Solution:-

Given $z^4q^2 - z^2p = 1$

The given equation can also be written as $(z^2q)^2 - (z^2p) = 1$

It is of the form $f(z^k p, z^k q)=0$, where k=2

Putting $Z = z^{k+1} = z^3$, we get

$$P = \frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial z^3}{\partial z} p = 3z^2 p$$

$\Rightarrow P/3=z^2p$

$$Q = \frac{\partial Z}{\partial y} = \frac{\partial Z}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial z^3}{\partial z} q = 3z^2 q$$

$\Rightarrow Q/3=z^2q$

Hence the given equation reduces to

$$\left(\frac{Q}{3}\right)^2 - \frac{P}{3} = 1$$

i.e., $Q^2 - 3P - 9 = 0$,

which is of the form $F(P, Q) = 0$.

Hence its complete solution is given by

$Z = ax + by + c$, where $b^2 - 3a - 9 = 0$.

Solving for b,

$$b = \pm\sqrt{3a + 9}$$

Hence the complete solution is

$$Z = ax \pm \left(\sqrt{3a + 9}\right)y + c$$

Sub $Z=z^3$

$$z^3 = ax \pm \left(\sqrt{3a + 9}\right)y + c$$

EXERCISES

Solve the following equations.

1. $x^2p^2 + y^2p^2 = z^2$

2. $z^2 (p^2+q^2) = x^2 + y^2$

3. $z^2 (p^2x^2 + q^2) = 1$

4. $2x^4p^2 - yzq - 3z^2 = 0$

5. $p^2 + x^2y^2q^2 = x^2 z^2$

6. $x^2p + y^2q = z^2$

7. $x^2/p + y^2/q = z$

8. $z^2 (p^2 - q^2) = 1$

9. $z^2 (p^2/x^2 + q^2/y^2) = 1$

10. $p^2x + q^2y = z$.

LAGRANGE'S LINEAR EQUATION:

Differential equations of the form $P(x, y, z)p + Q(x, y, z)q = R(x, y, z)$ ——— (1) are linear in *p* and *q*, also it is called Lagranges Linear differential equation.

To solve this equation,

let us consider the equations u = a and v = b, where a, b are arbitrary constants and u, v are functions of x, y, z.

Since *u* is a constant, we have

$du = 0$ (2)

But *u* as a function of *x, y, z*,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

Comparing (1) and (2), we have

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0 \qquad (3)$$

Similarly, $\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0$ (4)

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}} = \frac{dy}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}} = \frac{dz}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}}$$

(or) $$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (5)$$

Equations (5) represent a pair of simultaneous equations which are of the first order and of first degree.

Therefore, the two solutions of (5) are u = a and v = b. Thus,

$\varphi(u, v) = 0$is the required solution of(1).

Note:

To solve the Lagrange's equation, we have to form the subsidiary or auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

which can be solved either by the method of grouping (For easy problem) or by the method of multipliers(For difficult problem).

METHOD OF GROUPING

Problem :-21

Find the general solution of $px + qy = z$.

Solution:-

Given px+qy=z

It is of the form Pp+qQ=R, Here P=x, Q=y, R=z

The subsidiary equations is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$

First solution

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating,

log x = log y + log c_1, take exponential

$$x = c_1 y$$

$$\frac{x}{y} = c_1$$

Second solution

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating

$\log y = \log z + \log c_2$

$y = c_2 z$

$\frac{y}{z} = c_2$

Hence the required general solution is

$\varphi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$, where φ is arbitrary function

Problem:- 22

Solve p tan x + q tan y = tan z

Solution:-

Given p tan x + q tan y = tan z

It is of the form Pp+Qq=R here P=tan x, Q=tany , R=tanz

The subsidiary equations is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$

First solution

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

i.e., $\cot x\, dx = \cot y\, dy$

Integrating, $\log \sin x = \log \sin y + \log c_1$

i.e., $\sin x = c_1 \sin y$

Therefore, $\frac{\sin x}{\sin y} = c_1$

Second solution

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

i.e., $\cot y\, dy = \cot z\, dz$

Integrating, $\log \sin y = \log \sin z + \log c_2$

$\sin y = c_2 \sin z$

i.e., $\frac{\sin y}{\sin z} = c_2$

Hence the general solution is

$\varphi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$, where φ is arbitrary

Method of Multiplier

Problem:-23

Solve $(y - z)p + (z - x)q = x - y$

Solution:-

Given $(y - z)p + (z - x)q = x - y$

It is of the form Pp + Qq =R, here P=y-z, Q=z-x, R=x-y

The subsidiary equations is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$=*a(say)*

dx=a(y-z), dy=a(z-x), dz=a(x-y)

First solution

Consider the multiplier dx+dy+dz

dx+dy+dz=a(y-z)+a(z-x)+a(x-y)

=a{y-z+z-x+x-y}=a.0=0

dx+dy+dz=0

Integrate on both sides.

$\int dx + dy + dz = .c_1$

$x + y + z = c_1$ (1)

Second solution

consider multiplier xdx+ydy+zdz

xdx+ydy+zdz=xa(y-z)+ya(z-x)+za(x-y)

=a(xy-xz+yz-yx+zx-zy)=0

xdx+ydy+zdz=0

Integrating on both sides

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$

$x^2 + y^2 + z^2 = 2c_2 = c_3$ (2)

Hence from (1) and (2), the general solution is

$\varphi(x + y + z, x^2 + y^2 + z^2) = 0$, where φ is arbitrary function

Problem:- 24

Find the general solution of $(mz - ny)p + (nx - lz)q = ly - mx$

Solution:-

Given $(mz - ny)p + (nx - lz)q = ly - mx$

It is of the form Pp+Qq=R, here P=mz-ny, Q=nx-lz, R=ly-mx

The subsidiary equations is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e $\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$=a(say)

dx=a(mz-ny), dy=a(nx-lz), dz=a(ly-mx)

First solution

xdx+ydy+zdz=ax(mz-ny)+ya(nx-lz)+za(ly-mx)

=a{xmz-xny+ynx-ylz+zly-zmx}=0

xdx+ydy+zdz=0

Integrate on both sides

$\int xdx + ydy + zdz = c_1$

$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$

$x^2 + y^2 + z^2 = 2c_1 = c_2$ (1)

Second solution

Consider the multiplier ldx+mdy+ndz

ldx+mdy+ndz=la(mz-ny)+ma(nx-lz)+na(ly-mx)

=0

Integrate on both sides

$\int ldx + mdy + ndz = c_3$

$lx + my + nz = c_3$ (2)

Hence, the required general solution is

$\varphi(x^2 + y^2 + z^2, lx + my + nz) = 0$, where φ is arbitrary function.

Problem:-25

Solve $(x^2 - y^2 - z^2)\, p + 2xy\, q = 2xz$.

Solution:-

Given $(x^2 - y^2 - z^2)\, p + 2xy\, q = 2xz$.

It is of the form Pp+Qq=R, here $P=x^2-y^2-z^2$, $Q=2xy$, $R= 2xz$

The subsidiary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$

First solution (Method of gouping)

Taking the last two ratios,

$\frac{dy}{2xy} = \frac{dz}{2xz}$

i.e., $\frac{dy}{y} = \frac{dz}{z}$

Integrate on both sides

$\int \frac{dy}{y} = \int \frac{dz}{z}$

$\log y = \log z + \log c_1$

$y = c_1 z$

i.e., $\frac{y}{z} = c_1$ (1)

Second solution (Method of Multiplier)

$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = a(say)$----(2)

$dx=a(x^2-y^2-z^2)$, $dy=2axy$, $dz=2axz$

consider the multiplier xdx+ydy+zdz

$xdx+ydy+zdz=a\{x(x^2-y^2-z^2)+2xy^2+2xz^2\}$

$=ax\{x^2-y^2-z^2+2y^2+2z^2\}=ax(x^2+y^2+z^2)$

$(xdx+ydy+zdz)/x(x^2+y^2+z^2)=a$

Comparing with the last ratio, we get

$\frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}=\frac{dz}{2xz}$ [using equation (2)]

i.e., $\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}=\frac{dz}{z}$

Since $dx^2=2xdx$, $dy^2=2ydy$, $dz^2=2zdz$

$$\frac{dx^2+dy^2+dz^2}{x^2+y^2+z^2}=\frac{dz}{z}$$

$$\frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2}=\frac{dz}{z}$$

$\frac{ds}{s}=\frac{dz}{z}$ where $s=x^2+y^2+z^2$

$\log s=\log z+\log c_2$

$\log s=\log zc_2$

$\log(x^2+y^2+z^2)=\log z\ c_2$

$x^2+y^2+z^2=c_2 z$

i.e., $\frac{x^2+y^2+z^2}{z}=c_2$ (3)

From (1) and (3), the general solution is given by $\varphi\left(\frac{y}{z},\frac{x^2+y^2+z^2}{z}\right)=0$,

where φ is arbitrary function.

EXERCISES

Solve the following PDE

1. $px^2+qy^2=z^2$

2. $pyz+qzx=xy$

3. $xp-yq=y^2-x^2$

4. $y^2zp+x^2zq=y^2x$

5. z (x – y) = $px^2 – qy^2$

6. (a – x) p + (b – y) q = c – z

7. (y^2z p) /x + xzq = y^2

8. ($y^2 + z^2$) p – xyq + xz = 0

9. $x^2p + y^2q$ = (x + y) z

10. p – q = log(x+y)

11. (xz + yz)p + (xz– yz)q = x^2+y^2

12. (y – z)p– (2x + y)q = 2x +z

PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITH CONSTANTCOEFFICIENTS

Homogeneous Linear Equations with constant Coefficients.

A homogeneous linear partial differential equation of the n^{th} order is of the form $c_0 \frac{\partial^n z}{\partial x^n} + c_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \cdots + c_n \frac{\partial^n z}{\partial y^n} = F(x, y)$ (1)

Where$c_0, c_1, \dots, c_n$ are constants and F is a function of *x* and *y*. It is homogeneous because all its terms contain derivatives of the same order.

Equation (1) can be expressed as

$$\left(c_0 D^n + c_1 D^{n-1} D' + \cdots + c_n D'^n\right) z = F(x, y)$$

or $f(D, D')\, z = F(x, y)$ (2)

Where $\frac{\partial}{\partial x} \equiv D$ and $\frac{\partial}{\partial y} \equiv D$

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely, the complementary function and the particular integral.

The complementary function is the complete solution of$f(D, D')z = 0 \rightarrow$(3), Which must contain n arbitrary functions as the degree of the polynomial $f(D, D')$. The particular integral is the particular solution of equation (2).

Finding the complementary function

Let us now consider the equation $f(D, D')z = F(x, y)$

The auxiliary equation of (3) is obtained by replacing D by D' by 1.

$c_0 m^n + c_1 m^{n-1} + \cdots + c_n = 0$ (4)

Solving equation (4) for *m*, we get *n* roots. Depending upon the nature of the roots, the Complementary function is written as given below:

Roots of the auxiliary equation	Nature of the roots	Complementary function(C.F)
$m_1, m_2, m_3, \ldots, m_n$	distinct roots	$f_1(y+m_1x)+f_2(y+m_2x) +\ldots+f_n(y+m_nx)$.
$m_1 = m_2 = m$, $m_3, m_4, \ldots, m_n$	two equal roots	$f_1(y+m_1x)+xf_2(y+m_1x) + f_3(y+m_3x) +\ldots+f_n(y+m_nx)$.
$m_1 = m_2 = \ldots = m_n = m$	all equal roots	$f_1(y+mx)+xf_2(y+mx) + x^2f_3(y+mx) +\ldots +x^{n-1} f_n(y+mx)$

Finding the particular Integral

Consider the equation $f(D, D')z = F(x, y)$.

Now, the P.I is given by $\frac{1}{f(D,D')}F(x, y)$

Case (i):

When $F(x, y) = e^{ax+by}$

$$P.I = \frac{1}{f(D, D')} e^{ax+by}$$

Replacing D by *a* and D'by *b*, we have

$P.I = \frac{1}{f(a,b)} e^{ax+by}$, where $f(a, b) \neq 0$

Case (ii) :

When $F(x, y) = \sin(ax + by)(or) \cos(ax + by)$

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax + by)(or) \cos(ax + by)$$

Replacing $D^2 = -a^2$, $DD'^2 = -ab$ and $D' = -b^2$, we get

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by)(or) \cos(ax + by),$$

where $f(-a^2, -ab, -b^2) \neq 0$

Case (iii) :

When $F(x, y) = x^m y^n$,

$$P.I = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$$

Expand $[f(D, D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

Case (iv):

When $F(x, y)$ is any function of *x* and *y*.

$$P.I = \frac{1}{f(D, D')} F(x, y)$$

Resolve $\frac{1}{f(D,D')}$ into partial fractions considering $f(D, D')$ as a function of D alone.

Then operate each partial fraction on $F(x, y)$ in such a way that

$\frac{1}{D-mD'}F(x,y) = \int F(x, c - mx)dx,$

where *c* is replaced by y + mx after integration

Problem:-26

Solve$(D^3 - 3D^2D' + 4D'^3)$ z = e^{x+2y}

Solution:-

The auxiliary equation is m=$m^3 - 3m^2 + 4 = 0$

The roots are m = -1,2,2

Therefore, the C.F is $f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$.

$$P.I = \frac{e^{x+2y}}{D^3 - 3D^2D' + 4D'^3}$$ (Replace D by 1 and D'by2)

$$= \frac{e^{x+2y}}{1-3(1)(2)+4(2)^3}$$

$$P.I = \frac{e^{x+2y}}{27}$$

Hence, the solution is z = C.F. + P.I

i.e., $z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{e^{x+2y}}{27}$

Problem:- 27

Solve $(D^2 - 4DD' + 4D'^2)$ z = cos (x – 2y)

Solution:-

The auxiliary equation is $m^2 - 4m + 4 = 0$ Solving, we get m = 2,2.

Therefore, the C.F is $f_1(y+2x) + xf_2(y+2x)$.

$$\therefore P.I = \frac{1}{D^2 - 4DD' + 4D'^2}\cos(x - 2y)$$

Replacing D^2 by – 1, DD' by 2 and D'^2 by –4, we have

$$P.I = \frac{1}{(-1) - 4(2) + 4(-4)}\cos(x - 2y)$$

$$= -\frac{\cos(x-2y)}{25}$$

Hence, the solution is $z = f_1(y+2x) + xf_2(y+2x) - \frac{\cos(x-2y)}{25}$

Problem:- 28

Solve $(D^2 - 2DD')\,z = x^3y + e^{5x}$

Solution:-

The auxiliary equation is $m^2 - 2m = 0$.

Solving, we get $m = 0,2$.

Hence the C.F is $f_1(y) + f_2(y+2x)$.

$$P.I_1 = \frac{x^3y}{D^2 - 2DD'}$$

$$= \frac{1}{D^2\left(1 - \frac{2D'}{D}\right)}(x^3y)$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1}(x^3y)$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \cdots\right)(x^3y)$$

$$= \frac{1}{D^2}(x^3y) + \frac{2D'}{D}(x^3y) + \frac{4D'^2}{D^2}(x^3y) + \cdots$$

$$= \frac{1}{D^2}(x^3y) + \frac{2}{D}(x^3) + \frac{4}{D^2}(0) + \cdots$$

$$= \frac{1}{D^2}(x^3y) + \frac{2}{D}(x^3)$$

$$P.I_1 = \frac{x^5y}{20} + \frac{x^3}{60}$$

$$P.I_2 = \frac{e^{5x}}{D^2 - 2DD'}$$ (Replace D by 5 and D' by0)

$$= \frac{e^{5x}}{25}$$

Hence, the solution is $Z = f_1(y) + f_2(y+2x) + \frac{x^5 y}{20} + \frac{x^3}{60} + \frac{e^{5x}}{25}$

Problem:-29

Solve $\left(D^2 + DD' - 6D'^2\right) z = y\cos x$

Solution:-

The auxiliary equation is $m^2 + m - 6 = 0$.

Therefore, $m = -3, 2$.

Hence the C.F is $f_1(y - 3x) + f_2(y + 2x)$.

$$P.I = \frac{y\cos x}{D^2 + DD' - 6D'^2}$$

$$= \frac{y\cos x}{(D + 3D')(D - 2D')}$$

$$= \frac{1}{(D + 3D')} \frac{1}{(D - 2D')} y\cos x$$

$$= \frac{1}{(D+3D')} \int (c - 2x)\cos x\, dx, \text{ where } y = c - 2x$$

$$= \frac{1}{(D + 3D')} \int (c - 2x)\, d(\sin x)$$

$$= \frac{1}{(D+3D')} [(c - 2x)(\sin x) - (-2)(-\cos x)]$$

$$= \frac{1}{(D + 3D')} [y\sin x - 2\cos x]$$

$$= \int [(c + 3x)\sin x - 2\cos x] dx, \text{ where } y = c + 3$$

$$= \int (c + 3x)\, d(-\cos x) - 2\int \cos x\, dx$$

$$= (c + 3x)(-\cos x) - (3)(-\sin x) - 2\sin x$$

$$= -y\cos x + \sin x$$

Hence the complete solution is

$z = f_1(y - 3x) + f_2(y + 2x) - y\cos x + \sin x$

Problem:-30

Solve $r - 4s + 4t = e^{2x+y}$

Solution:-

Given equation is $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x\,\partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

i.e., $\left(D^2 - 4DD' + 4D'^2\right)z = e^{2x+y}$

The auxiliary equation is $m^2 - 4m + 4 = 0$. Therefore, $m = 2,2$

Hence the C.F is $f_1(y + 2x) + xf_2(y + 2x)$

$$P.I = \frac{e^{2x+y}}{D^2 - 4DD' + 4D'^2}$$

Since $D^2 - 4DD' + 4D'^2$ for D = 2 and D' = 1, we have to apply the general rule.

$$P.I = \frac{e^{2x+y}}{(D - 2D')(D - 2D')}$$

$$= \frac{1}{(D - 2D')}\frac{1}{(D - 2D')}e^{2x+y}$$

$$= \frac{1}{(D-2D')}\int e^{2x+c-2x}\,dx, \text{ where } y = c - 2x$$

$$= \frac{1}{(D-2D')}\int e^{c}dx$$

$$= \frac{1}{(D - 2D')}e^{c}.x$$

$$= \frac{1}{(D - 2D')}x\,e^{y+2x}$$

$\int x\,e^{c-2x+2x}\,dx$ where $y = c - 2x$.

$$= \int x\,e^{c}\,dx$$

$$= e^{c}\left(\frac{x^2}{2}\right)$$

$$= \frac{x^2 e^{y+2x}}{2}$$

Hence the complete solution is

$$z = f_1(y + 2x) + xf_2(y + 2x) + \frac{x^2 e^{y+2x}}{2}$$

NON – HOMOGENEOUS LINEAREQUATIONS

Let us consider the partial differential equation

$$f(D, D')z = F(x, y) \qquad (1)$$

If $f(D, D')$ is not homogeneous, then (1) is a non–homogeneous linear partial differential equation. Here also, the complete solution = C.F + P.I.

The methods for finding the Particular Integrals are the same as those for homogeneous linear equations.

But for finding the C.F, we have to factorize $f(D, D')$ into factors of the form $D - mD' - c$.

Consider now the equation

$$(D - mD' - c)z = 0 \qquad (2)$$

This equation can be expressed as

$$p - mq = cz \qquad (3),$$

which is in Lagrangian form.

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz} \qquad (4)$$

The solutions of (4) are $y + mx = a$ and $z = be^{cx}$.

Taking $b = f(a)$, we get $z = be^{cx} f(y + mx)$ as the solution of (2).

Note:

If $[(D - m_1D' - c_1)(D - m_2D' - c_2) \dots (D - m_nD' - c_n)]z = 0$

is the partial differential equation, then its complete solution is

$z = e^{c_1x}f_1(y + m_1x) + e^{c_2x}f_2(y + m_2x) + \cdots + e^{c_nx}f_n(y + m_nx)$

In the case of repeated factors, the equation $(D - m_nD' - c_n)z=0$ has a complete solution

$z = e^{cx}f_1(y + mx) + xe^{cx}f_2(y + mx) + \cdots + x^{n-1}e^{cx}f_n(y + mx)$

Problem:-31

Solve $(D - D' - 1)\,(D - D' - 2)z = e^{2x-y}$

Solution:-

Here, $m_1 = 1, m_2 = 1, c_1 = 1, c_2 = 2.$

Therefore, the C.F is $e^{x}f_1(y + x) + e^{2x}f_2(y + x)$

$$P.I = \frac{e^{2x-y}}{(D - D' - 1)\,(D - D' - 2)}$$

Put $D = 2, D' = -1.$

$$= \frac{e^{2x-y}}{(2 - (-1) - 1)\,(2 - (-1) - 2)}$$

$$= \frac{e^{2x-y}}{2}$$

Hence the solution is $z = e^{x}f_1(y + x) + e^{2x}f_2(y + x) + \frac{e^{2x-y}}{2}$

Problem:- 32

Solve $(D^2 - DD' + D' - 1)z = \cos(x + 2y)$

Solution:-

The given equation can be rewritten as

$(D - D' + 1)(D - 1)z = \cos(x + 2y)$

Here, $m_1 = 1, m_2 = 0, c_1 = -1, c_2 = 1$

Therefore, the C.F is $e^{-x}f_1(y+x) + e^{x}f_2(y)$

$$P.I = \frac{1}{(D^2 - DD' + D' - 1)}\cos(x+2y)$$

Put $D^2 = -1, DD' = -2, D' = -4$

$$= \frac{1}{-1-(-2)+D'-1}\cos(x+2y)$$

$$= \frac{1}{D'}\cos(x+2y)$$

$$= \frac{\sin(x+2y)}{2}$$

Hence the solution is $z = e^{-x}f_1(y+x) + e^{x}f_2(y) + \frac{\sin(x+2y)}{2}$

Problem:- 33

Solve$[(D + D' - 1)(D + 2D' - 3)]z = e^{x+2y} + 4 + 3x + 6y$

Solution:-

$$\text{Here, } m_1 = -1, m_2 = -2, c_1 = 1, c_2 = 3$$

Hence the C.F is $e^{x}f_1(y-x) + e^{3x}f_2(y-2x)$

$$P.I_1 = \frac{e^{x+2y}}{(D + D' - 1)(D + 2D' - 3)}$$

Put $D = 1, D' = 2$

$$= \frac{e^{x+2y}}{(1+2-1)(1+4-3)}$$

$$P.I_1 = \frac{e^{x+2y}}{4}$$

$$P.I_2 = \frac{1}{(D + D' - 1)(D + 2D' - 3)}(4 + 3x + 6y)$$

$$= \frac{1}{3\left[1 - (D + D')\left(1 - \frac{D+2D'}{3}\right)\right]}(4 + 3x + 6y)$$

$$= \frac{1}{3}[1 - (D + D')]^{-1}\left(1 - \frac{D + 2D'}{3}\right)^{-1}(4 + 3x + 6y)$$

$$= \frac{1}{3}\left[1 + (D + D') + (D + D')^2 + \cdots\right]\left[1 + \left(\frac{D + 2D'}{3}\right) + \left(\frac{D + 2D'}{3}\right)^2 + \cdots\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[(1) + \frac{4}{3}(D) + \frac{5}{3}(D') \ldots\right](4 + 3x + 6y)$$

$$= \frac{1}{3}\left[(4 + 3x + 6y) + \frac{4}{3}(3) + \frac{5}{3}(6)\right]$$

P. $I_2 = x + 2y + 6$

Hence the complete solution is

$$z = e^x f_1(y - x) + e^{3x} f_2(y - 2x) + \frac{e^{x+2y}}{4} + x + 2y + 6$$

EXERCISES

Solve the following homogeneous Equations.

1. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x + y)$

2. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

3. $(D^2 + 3DD' + 2D'^2)\, z = x + y$

4. $(D^2 - DD' + 2D'^2)\, z = xy + e^x . \cosh y$

$$\left\{\text{Hint: } e^x \cosh y = e^x\left(\frac{e^y + e^{-y}}{2}\right) = \left(\frac{e^{x+y} + e^{x-y}}{2}\right)\right\}$$

5. $(D^3 - 7DD'^2 - 6D'^3)\, z = \sin(x+2y) + e^{2x+y}$

6. $(D^2 + 4DD' - 5D'^2)\, z = 3e^{2x-y} + \sin(x - 2y)$

7. $(D^2 - DD' - 30D'^2)\, z = xy + e^{6x+y}$

8. $(D^2 - 4D'^2)\, z = \cos 2x . \cos 3y$

9. $(D^2 - DD' - 2D'^2)\, z = (y - 1)e^x$

10. $4r + 12s + 9t = e^{3x-2y}$

Solve the following non – homogeneous equations.

1. $(2DD' + D'^2 - 3D')\, z = 3\cos(3x - 2y)$

2. $(D^2 + DD' + D' - 1)\, z = e^{-x}$

3. $r - s + p = x^2 + y^2$

4. $(D^2 - 2DD' + D'^2 - 3D + 3D' + 2)z = (e^{3x} + 2e^{-2y})^2$

5. $(D^2 - D'^2 - 3D + 3D')\, z = xy + 7.$

LECTURE NOTES

UNIT-V

APPLICATION PARTIAL DIFFERENTIAL EQUATIONS

INTRODUCTION

In many real life problem which is represented in ordinary or partial differential equation. We required a solution which satisfies some specified conditions *called boundary conditions.*

Any differential equation with these boundary condition *is called boundary value problem.*

In case of PDE we get solution involved in arbitrary constants and arbitrary function. Hence it is difficult for us to adjust these conditions and function so as to get an optimal solution satisfying the boundary conditions. To over come these we adopt *method of separation of variables* for solving linear PDE as it satisfy all or some boundary conditions.

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER

The general form of linear partial differential equation of second order is given by $A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$ ......(1)

(i.e) $Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=0$, where A, B, C, D, E, F are general functions of x and y.

ELLIPTIC EQUATION

The equation $A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$ of second order is called elliptic at (x,y) if $B^2-4AC<0$.

[where A, B, C, D, E, F are functions of x and y]

Example:

(i) Laplace equation in two dimension $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

A=1, B=0, C=1

$B^2-4AC=0^2-4(1)(1)=-4<0$

Therefore the given equation is *elliptic*

(ii) Poisson equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f$

A=1, B=0, C=1

$B^2-4AC=0^2-4(1)(1)=-4<0$

Therefore the given equation is *elliptic*

PARABOLIC EQUATION

The equation $A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$ of second order is called parabolic at (x,y) if $B^2-4AC=0$.

[where A, B, C, D, E, F are functions of x and y]

Example:-

One dimension heat flow equation $\frac{\partial^2 u}{\partial x^2}=\alpha\frac{\partial u}{\partial t}$

$$\frac{\partial^2 u}{\partial x^2}-\alpha\frac{\partial u}{\partial t}=0$$

A=1, B=0, C=0,

$B^2-4AC=0^2-4(1)(0)=0$

Therefore the given equation is *parabolic*

HYPERBOLIC EQUATION

The equation $A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$ of second order is called hyperbolic at (x,y) if $B^2-4AC>0$.

[where A, B, C, D, E, F are functions of x and y]

Example:-

One dimensional wave equation $\frac{\partial^2 u}{\partial x^2} = a\frac{\partial^2 u}{\partial t^2}$

i.e $\frac{\partial^2 u}{\partial x^2} - a\frac{\partial^2 u}{\partial t^2} = 0$

A=1, B=0, C=-a

$B^2-4AC=0^2-4(1)(-a)=4a>0$

Therefore the given equation is *hyperbolic*

Problem:-01

Classify the following equation $x\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$

Solution:-

Given $x\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$

A=x, B=0, C=1

$B^2-4AC=0^2-4(x)(1)=-4x$

If -4x<0, i.e 4x>0, this implies x>0, then given equation is *elliptic*

If -4x=0, i.e 4x=0, this implies x=0, then given equation is *parabolic*

If -4x>0, i.e 4x<0, this implies x<0, then given equation is *hyperbolic*.

Problem:-02

Classify the following equation $x^2 f_{xx} + (1-y^2) f_{yy} = 0$, where -1<y<1 and

$-\infty < x < \infty$

Solution

Given $x^2 f_{xx} + (1-y^2) f_{yy} = 0$, where -1<y<1 and $-\infty < x < \infty$

$A=x^2$, B=0, $C=1-y^2$

$B^2-4AC=0^2-4(x^2)(1-y^2)=-4x^2(1-y^2)<0$

Therefore the given equation is *elliptic* except x=0

when *x=0*, then $B^2-4AC=0$, therefore the given equation is *parabolic*.

Problem:-03

Classify the following equation $x\ f_{xx} + y f_{yy} = 0$, where x>0, y>0

Solution

Given $x\ f_{xx} + y f_{yy} = 0$, where x>0, y>0

A=x, B=0, C=y

$B^2-4AC=0^2-4(x)(y)=-4xy<0$

Therefore the given equation is *elliptic*.

Problem:-04

Classify the following equation $f_{xx} - 2 f_{xy} = 0$, where x>0, y>0

Solution

Given $f_{xx} - 2 f_{xy} = 0$, where x>0, y>0

A=1, B=-2, C=0

$B^2-4AC=(-2)^2-4(1)(0)=4>0$

Therefore the given equation is *hyperbolic*.

Problem:-05

Classify the following equation $f_{xx} - 2 f_{xy} + f_{yy} = 0$, where x>0, y>0

Solution

Given $f_{xx} - 2 f_{xy} + f_{yy} = 0$, where x>0, y>0

A=1, B=-2, C=1

$B^2-4AC=(-2)^2-4(1)(1)=0$

Therefore the given equation is *parabolic.*

Problem:-06

Classify the following equation $f_{xx}+2f_{xy}+4f_{yy}=0$, where x>0, y>0

Solution

Given $f_{xx}+2f_{xy}+4f_{yy}=0$, where x>0, y>0

A=1, B=2, C=4

B^2-4AC=$(2)^2$-4(1)(4)=-12<0

Therefore the given equation is *elliptic*.

Problem:-07

Classify the nature of the following equation $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0$

Solution:-

The given equation is $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0$ ....(1)

We know that the general form of linear second order partial differential equation is given by $A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+D\frac{\partial u}{\partial x}+E\frac{\partial u}{\partial y}+Fu=0$

........(2)

By comparing equations (1) and (2) we get the following A=1, B=2, C=1, D=0, E=0, F=0.

Now we calculate B^2-4AC=$(2)^2$-4(1)(1)=0 for all values of x and y.

Therefore the given equation (1) is *parabolic at all points of x and y.*

In other wards we can say that equation (1) represents parabolic equation over the XY-plane.

Problem:-08

Classify the nature of the following equation $x^2 f_{xx}+(1-y^2)f_{yy}=0$

Solution:-

The given equation is $x^2 f_{xx}+(1-y^2)f_{yy}=0$ ....(1)

We know that the general form of linear second order partial differential equation is given by $A\frac{\partial^2 f}{\partial x^2}+B\frac{\partial^2 f}{\partial x\partial y}+C\frac{\partial^2 f}{\partial y^2}+D\frac{\partial f}{\partial x}+E\frac{\partial f}{\partial y}+Ff=0$ ....(2)

By comparing equations (1) and (2) we get the following $A=x^2$, $B=0$, $C=1-y^2$, $D=0$, $E=0$, $F=0$.

Now we calculate $B^2-4AC=(0)^2-4(x^2)(1-y^2)=4x^2(y^2-1)$.

Since the B^2-4AC value is not just a number here, it involves variables x and y. So we have to consider the following case to classify the nature of the equation.

(i) Suppose if we assume that $B^2-4AC<0$

$\Rightarrow 4x^2(y^2-1)<0$, clearly 4, x^2 in this product are always positive.

Therefore y^2-1 must be negative number, (i.e) $y^2-1<0$

$\Rightarrow y^2<1$, (i.e) $-1<y<1$.

Thus $B^2-4AC<0$, only when $-1<y<1$ and $x\neq 0$(Since when $x=0$, we get $B^2-4AC=0$)

Hence equation (1) represent elliptic equation in the region where $x\neq 0$ and $-1<y<1$.

(ii) Suppose if we assume that $B^2-4AC=0$

$\Rightarrow 4x^2(y^2-1)=0$, clearly $x=0$ (or) $y^2-1=0$.

(i.e) $x=0$, and $y=+1$, $y=-1$.

Thus $B^2-4AC=0$, only when $x=0$, $y=-1$, $y=1$

Hence equation (1) represent parabolic equation when (x,y) lies on the lines $x=0$, $y=-1$ and $y=1$.

(iii) Suppose if we assume that $B^2-4AC>0$

$\Rightarrow 4x^2(y^2-1)>0$, clearly 4, x^2 in this product are always positive.

Therefore y^2-1 must be positive number, (i.e) $y^2-1>0$

$\Rightarrow y^2>1$, (i.e) $-\infty<y<-1$ and $1<y<\infty$.

Thus $B^2-4AC>0$, only when $-\infty<y<-1$ and $1<y<\infty$ and $x\neq 0$(Since when $x=0$, we get $B^2-4AC=0$)

Hence equation (1) represent hyperbolic equation in the region $-\infty < y < -1$ and $1 < y < \infty$. where $x \neq 0$.

Problem:-09

Classify the nature of the following equation

$u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = \sin(x+y)$

Solution:-

The given equation is $u_{xx} + 4u_{xy} + (x^2 + 4y^2)u_{yy} = \sin(x+y)$.....(1)

We know that the general form of linear second order partial differential equation is given by $A\frac{\partial^2 f}{\partial x^2} + B\frac{\partial^2 f}{\partial x \partial y} + C\frac{\partial^2 f}{\partial y^2} + D\frac{\partial f}{\partial x} + E\frac{\partial f}{\partial y} + Ff = 0$ ....(2)

By comparing equations (1) and (2) we get the following A=1, B=4, $C=x^2+4y^2$, D=0, E=0, F=0.

Now calculate $B^2-4AC=(4)^2-4(1)(x^2+4y^2)=16-4x^2-16y^2$.

Since the B^2-4AC value is not just a number here, it involves variables x and y. So we have to consider the following case to classify the nature of the equation.

(i) Suppose if we assume that $B^2-4AC<0$

$\Rightarrow 16-4x^2-16y^2<0$, To find (x,y) which satisfies this condition, we proceed as follow.

$\Rightarrow 16<4x^2+16y^2$

$\Rightarrow 1<(4x^2+16y^2)/16$, (i.e) $1< x^2/4+y^2/1$

$\Rightarrow \frac{x^2}{4} + \frac{y^2}{1} > 1$

Thus $B^2-4AC<0$, only when $\frac{x^2}{4} + \frac{y^2}{1} > 1$

Hence equation (1) represent elliptic equation in the region $\frac{x^2}{4} + \frac{y^2}{1} > 1$.

(ii) Suppose if we assume that $B^2-4AC=0$

$\Rightarrow 16-4x^2-16y^2=0$, To find (x,y) which satisfies this condition, we proceed as follow.

$\Rightarrow 16=4x^2+16y^2$

$\Rightarrow 1=(4x^2+16y^2)/16$, (i.e) $1= x^2/4+y^2/1$

$$\Rightarrow \frac{x^2}{4}+\frac{y^2}{1}=1$$

Thus $B^2-4AC=0$, only when $\frac{x^2}{4}+\frac{y^2}{1}=1$

Hence equation (1) represent parabolic equation in the region $\frac{x^2}{4}+\frac{y^2}{1}=1$.

(iii) Suppose if we assume that $B^2-4AC>0$

$\Rightarrow 16-4x^2-16y^2>0$, To find (x,y) which satisfies this condition, we proceed as follow.

$\Rightarrow 16>4x^2+16y^2$

$\Rightarrow 1>(4x^2+16y^2)/16$, (i.e) $1> x^2/4+y^2/1$

$$\Rightarrow \frac{x^2}{4}+\frac{y^2}{1}<1$$

Thus $B^2-4AC>0$, only when $\frac{x^2}{4}+\frac{y^2}{1}<1$

Hence equation (1) represent hyperbolic equation in the region $\frac{x^2}{4}+\frac{y^2}{1}<1$.

EXERCISE

Classify the nature of the following partial differential equations

1. $(1+x)u_{xx}-2(x+2)u_{xy}+(x+3)u_{yy}=0$
2. $xu_{xx}+yu_{yy}=0, \quad x>0, y>0$
3. $f_{xx}-2f_{xy}=0,$
4. $f_{xx}+2f_{xy}+4f_{yy}=0,$
5. $f_{xx}-2f_{xy}+f_{yy}=0.$

METHOD OF SEPARATION OF VARIABLES

Let Z be dependent variable on x & y, where x & y are independent variables.

We assume the solution to be the product of two variable function, one function in x alone and another in y alone.

Thus the solution of PDE is converted to solution of ODE.

Problem:-1

Using the method of separation of variables solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where

$u(x,0)=6e^{-3x}$

Solution:-

Given $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ -------(1)

Here u is a function x and t

Let u=X(x)T(t)--------(2) be the solution of the given differential equation, where X is a function of x only and T is function t only.

Differentiate (2) partially w.r.t x and t, we get

$\frac{\partial u}{\partial x} = X^{'}T$ -------(3)

$\frac{\partial u}{\partial t} = XT^{'}$ --------(4)

Sub (2),(3) and (4) in (1), we get

$X'T=2XT'+XT$

i.e $X'T=X(2T'+T)$

Separating the variables , we get

$\frac{X^{'}}{X} = \frac{2T^{'}+T}{T} = K$ (constant)

$\frac{X'}{X} = K$	$\frac{2T' + T}{T} = K$
$X' - KX = 0$	$2T' + T = KT$
Solution	$2T' = KT - T$
$\frac{dX}{dx} = KX$	$T' = T(K-1)/2$
$\frac{dX}{X} = Kdx$	**Solution**
Integrating on both sides, we get	$\frac{dT}{dt} = (K-1)T/2$
Log X=Kx+log a	$\frac{dT}{T} = (K-1)/2dt$
$X=e^{Kx+\log a}$	Log T=(K-1)t/2+logb
$X=e^{Kx}a$	$T=e^{(K-1)t/2+\log b}$
$X=ae^{Kx}$	$T=be^{(K-1)t/2}$

Therefore u=XT

$u= ae^{Kx}\ be^{(K-1)t/2}$

$u(x,t)=abe^{Kx}e^{(K-1)t/2}$---------(5)

Putting t=0 in (5) we get

$u(x,0)=abe^{KX}$----------(6)

But $u(x,0)=6e^{-3x}$------(7)

From (6) and (7),we get

ab=6, K=-3------(8)

Sub (8) in (5), we get

$u(x,t)=6e^{-3x}e^{(-4)t/2}$

$=6e^{-3x}e^{-2t}$

$u(x,t)=6e^{-(3x+2t)}$

Which is the required solution.

Problem:-02

Using the method of separation of variable solve $x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0$

Solution

Given $2x\frac{\partial z}{\partial x} - 3y\frac{\partial z}{\partial y} = 0$-------(A)

Here z is a function x and y

Let z=X(x)Y(y)--------(1) be the solution of the given differential equation, where X is a function of x only and Y is function y only.

Differentiate (2) partially w.r.t x and y, we get

$\frac{\partial z}{\partial x} = X'Y$ & $\frac{\partial z}{\partial y} = XY'$-----(2)

Sub (2) in (A), we get

2xX 'Y-3y XY '=0

i.e 2xX 'Y=3yXY'

Separating the variables , we get

$\frac{2xX'}{X} = \frac{3yY}{Y} = K$ (constant)

$\frac{2xX'}{X} = K$ $2xX' - KX = 0$ $(2xD - K)X = 0$ D'=d/dx **Solution** This is an ODE with variable coefficient Sub $x=e^z$ =>z=log x xD=D' , where D'=d/dz (2D'-K)X=0 $2\frac{dX}{dz} = KX$	$\frac{3yY'}{Y} = K$ $3yY' = KY$ $(3yD - K)Y = 0$ where D=d/dy **Solution** This is an ODE with variable coefficient Sub $y=e^z$=>z=logy yD=D', where D'=d/dz $(3D' - K)Y = 0$ $3\frac{dY}{dz} = KY$

$2\frac{dX}{X} = Kdz$	$3\frac{dY}{Y} = Kdz$
Integrating on both sides	Integrating on both sides
2logX=Kz+c	3Log Y=Kz+d
logX=Kz/2+c/2	logY=Kz/3+b/3
$X=e^{(K/2)z+c/2}$	$Y=e^{Kz/3+b/3}$
$X=e^{(K/2)\log x}\ e^{c/2}$	$Y=e^{Kz/3}\ e^{b/3}$
$X=x^{k/2}C_1$	$Y= C_2e^{(K/3)\log y}$
$X= C_1x^{k/2}$	$Y= C_2y^{(K/3)}$

Therefore u=XY

$$u= C_1x^{k/2}\ C_2y^{(K/3)}$$

$$u(x,y)= C_1\ C_2\ x^{k/2}\ y^{(K/3)}$$

Which is the required solution.

Problem:-3

Using the method of separation of variables solve $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$

where $u(0,y)=8e^{-3y}$

Solution:-

Given $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$ -------(1)

Here u is a function x and t

Let u=X(x)Y(y)--------(2) be the solution of the given differential equation, where X is a function of x only and Y is function y only.

Differentiate (2) partially w.r.t x and y, we get

$\frac{\partial u}{\partial x} = X'Y$ -------(3)

$\frac{\partial u}{\partial y} = XY'$ --------(4)

Sub (3) and (4) in (1), we get

X ' Y=4 XY '

Separating the variables , we get

$\frac{X'}{X} = \frac{4Y'}{Y} = K$ (constant)

$\frac{X'}{X} = K$ $X' - KX = 0$ **Solution** $\frac{dX}{dx} = KX$ $\frac{dX}{X} = Kdx$ Integrating on both sides, we get Log X=Kx+log a $X=e^{Kx+\log a}$ $X=e^{Kx}a$ $X=ae^{Kx}$	$\frac{4Y'}{Y} = K$ $4Y' - KY = 0$ **Solution** $4\frac{dY}{dy} = KY$ $4\frac{dY}{Y} = Kdy$ 4Log Y=Ky+logb $\log Y^4=Ky+\log b$ $Y^4=e^{Ky+\log b}$ $Y^4=e^{Ky}\,b$ $Y=e^{(K/4)y}b^{1/4}$ $Y=e^{(K/4)y}c$, where $c=b^{1/4}$

Therefore u=XT

$u= ae^{Kx}\,ce^{(K/4)y}$

$u(x,t)=ace^{Kx}e^{(K/4)y}$--------(5)

Putting x=0 in (5) we get

$u(0,y)=ace^{(K/4)y}$---------(6)

But $u(0,y)=8e^{-3y}$------(7)

From (6) and (7),we get

ac=8, K=-12------(8)

Sub (8) in (5), we get

$u(x,t)=8e^{-12x}e^{(-12/4)y}$

$=8e^{-12x}e^{-3y}$

$u(x,t)=8e^{-(12x+3y)}$

Which is the required solution.

ONE DIMENSIONAL WAVE EQUATION

Consider the string is stretched and fastened to two points l apart.

Let T denotes the tension, m denoted the mass of the string.

The one dimensional wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } c^2 = \frac{T}{m}$$

The possible solutions are given by

(i) $(c_1e^{px}+c_2e^{-px})(c_3e^{cpt}+c_4e^{cpt})$

(ii) $(c_5\cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt)$

(iii) $(c_9x+c_{10})(c_{11}t+c_{12})$

We have to select the suitable solution which is consistent with physical nature of the problem , as we are dealing with problem on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric function, Therefore the best suitable solution for wave analysis is solution (ii).

i.e $u(x,t) = (c_5\cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt)$

Note

The boundary conditions are

(i) $u(0,t)$ denotes displacement (or vibration) at x=0 at any time t.

(ii) $u(l,t)$ denotes the displacement (or vibration) at x=l at any time t.

The Initial conditions are

(iii) $u(x,0)$ denotes the initial shape of the string at time t=0.

(iv) $\frac{\partial u}{\partial t}$ at time t=0., it is the initial velocity of the problem.

Almost in all the problems related to one dimensional wave equation two boundary conditions are zero .

Only one initial condition will be given in the problem, the other is assumed to be zero.

SOLUTION OF WAVE EQUATION BY THE METHOD SEPARATION OF VARIABLE

We know that one dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ ------(1)

let y=y(x,t)=X(x)T(t)-------(2) be the solution of the given equation , where X is a function of 'x' only and T is a function of 't' only.

Differentiate (2) partially w.r.t 'x' and 't', two times , we get

$\frac{\partial^2 y}{\partial x^2} = X''T$ and $\frac{\partial^2 y}{\partial t^2} = XT''$

Substituting these values in equation (1), we get

$XT''=a^2X''T$

$\frac{X''}{X} = \frac{T''}{a^2 T} = K(say)$

By separating the variable

$\frac{X''}{X} = K$ X''-KX=0-----(3) $(D^2-K)X=0$	$\frac{T''}{a^2 T} = K$ $T''-Ka^2T=0$ --------(4) $(D^2-Ka^2)T=0$

The equations (3) and (4) are ordinary differential equation, the solution of which depends on the values of K are three cases arises.

Case(i)

Let K be positive, i.e $K=p^2$ {Here p^2 is always positive whether p is +ve or -ve}

(3)=>X"-p^2X=0 (D^2-p^2)X=0 The auxiliary equation is $M^2-p^2=0$ M=+p,-p P.I=0 $X=C_1e^{px}+C_2e^{-px}$ -----(5)	(4)=>T"-p^2a2T=0 (D^2-a^2 p^2)X=0 The auxiliary equation is $M^2-a^2p^2=0$ M=+ap,-ap P.I=0 $T=C_3e^{apt}+C_4e^{-apt}$-------(6)

$y(x,t)=(C_1e^{px}+C_2e^{-px})(C_3e^{apt}+C_4e^{-apt})$

Case(ii)

Let K be a negative, i.e K=-p^2 {Here p^2 is always positive whether p is +ve or -ve}

(3)=>X"+p^2X=0 (D^2+p^2)X=0 The auxiliary equation is $M^2+p^2=0$ M=+ip,-ip $X=C_5cospx+C_6sinpx$ -----(7)	(4)=>T"+p^2a2T=0 (D^2+a^2 p^2)X=0 The auxiliary equation is $M^2+a^2p^2=0$ M=+iap,-iap $T=C_7cospat+C_8sinpat$-------(8)

Substitute equation (7) and (8) in (2), we get

$y(x,t)=(C_4cospx+C_6sinpx)(C_7cospat+C_8sinpat)$

Case(iii)

Let K=0

(3)=>X"=0 D^2X=0 The auxiliary equation is $M^2=0$ M=0,0 $X=(C_9x+C_{10})e^{0x}$-----(9)	(4)=>T" =0 D^2T=0 The auxiliary equation is $M^2=0$ M=0,0 $T=(C_{11}t+C_{12})e^{0t}$-------(10)

substitute equation (9) and (10) in (2), we get

$y(x,t)=(C_9x+C_{10})(C_{11}t+C_{12})$

Thus depending upon the value of K, the various possible solution of the wave equation are

$y(x,t)=(C_1e^{px}+C_2e^{-px})(C_3e^{apt}+C_4e^{-apt})$-------(11)

$y(x,t)=(C_4\cos px+C_6\sin px)(C_7\cos pat+C_8\sin pat)$-------(12)

$y(x,t)=(C_9x+C_{10})(C_{11}t+C_{12})$-------(13)

Now let us choose the solution which satisfy the boundary conditions of the given problem.

In general, in the problems of vibration of strings the two boundary or end conditions are $y(0,t)=0$ and $y(l,t)=0$ (always fixed), because at the ends $x=0$ and $x=l$, the string is fixed.

Hence to apply the above two boundary condition in the above solution, we have to select the correct one which is suitable for our problem.

(I) Consider the solution (11)

$y(x,t)=(C_1e^{px}+C_2e^{-px})(C_3e^{apt}+C_4e^{-apt})$ ---(11)

Apply the condition $y(0,t)=0$ in (11)

sub $x=0$ in (11)

$y(0,t)=(C_1e^0+C_2e^0)(C_3e^{apt}+C_4e^{-apt})$

$0=(C_1+C_2)(C_3e^{apt}+C_4e^{-apt})$

$0=(C_1+C_2)$-----(A) ($C_3e^{apt}+C_4e^{-apt}$ is not equal to zero)

Apply the condition $y(l,t)=0$ in (11)

sub $x=l$ in (11)

$y(l,t)=(C_1e^{pl}+C_2e^{-pl})(C_3e^{apt}+C_4e^{-apt})$

$0=(C_1e^{pl}+C_2e^{-pl})$ ------(B) ($C_3e^{apt}+C_4e^{-apt}$ is not equal to zero)

Solving (A), (B), we get

$C_1=0$ and $C_2=0$

Substituting in (11), we get $y(x,t)=0$.

(II) Consider the solution (13)

$y(x,t)=(C_9x+C_{10})(C_{11}t+C_{12})$--(13)

Apply the condition $y(0,t)=0$ in (13)

sub $x=0$ in (13)

$y(0,t)=(C_9.0+C_{10})(C_{11}t+C_{12})$

$0=(C_{10})(C_{11}t+C_{12})$

$0=C_{10}$----(D) ($C_{11}t+C_{12}$ is non zero)

Apply the condition $y(l,t)=0$ in (13)

sub $x=l$ in (13)

$y(l,t)=(C_9l+C_{10})(C_{11}t+C_{12})$

$0=(C_9l)(C_{11}t+C_{12})$ ($C_{10}=0$)

$0=C_9l$ ($C_{11}t+C_{12}$ is non zero)

$0=C_9$-----(E) (l is non zero)

Substituting (D) and (E) in (13), we get

$y(x,t)=0$,which is again a trivial solution

therefore (13) is also not the correct solution

Hence the correct solution is

$y(x,t)=(C_4\cos px+C_6\sin px)(C_7\cos pat+C_8\sin pat)$

Note:-

Simply for the vibration of string problem y must be periodic function of x and t. So we choose the solution which contains the trigonometric function.

$y(x,t)=(C_4\cos px+C_6\sin px)(C_7\cos pat+C_8\sin pat)$

Here *sin and cos* are periodic functions.

Boundary Value Problem

The boundary value problem has conditions specified at the extremes ("boundaries") of the independent variables in the given differential equation.

EXAMPLE:-

y''+ay'+by=c, with y(t)=d, y'(s)=e, assume that x defined on [t,s]

Here y is dependent variable and x is independent variable.

The conditions y(t)=d, y'(s)=e are specified at the extremes namely s and t.

Therefore it is a boundary value problem.

Initial Value Problem

The initial value problem has all conditions specified at the same value (that value is the lower boundary of the domain, thus the term "initial" value) of the independent variables in the given differential equation.

EXAMPLE:-

y''+ay'+by=c, with y(t)=d, y'(t)=e, assume that x defined on [t,s]

Here y is dependent variable and x is independent variable.

The conditions y(t)=d, y'(t)=e are specified at the same value t of independent variable x.

Therefore it is a initial value problem.

In a differential equation, we get general solution which contains arbitrary constants and then we determine these constants from the given initial values. This type of problems *are called initial value problem.*

A solution of DE which satisfies some specified conditions at the boundaries are *called boundary conditions.*

Any D.E together with these boundary conditions are *called boundary value problem.*

Problem:-01

A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form y=a sin ($\frac{\pi x}{l}$) from which it is released at time t=0. Show that the displacement of any point at a distance x from one end at time t is given by y(x,t)= a sin ($\frac{\pi x}{l}$) cos ($\frac{\pi ct}{l}$)

Solution:-

The one dimensional wave equation is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T}{m}$ --(1)

Boundary conditions are

y (0 , t)= 0------(2)

y (l , t)= 0------(3)

Initial conditions are

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ --------(4)

y (x, 0) = a sin ($\frac{\pi x}{l}$) ------(5)

The solution of equation (1) is given by

y (x , t) = ($c_1$cos px +c_2 sin px) (c_3 cos cpt +c_4 sin cpt) --------(6)

Apply boundary condition (2) in equation (6), we have

i.e Substitute x=0 in the equation (6), we have

y (0 , t) = ($c_1$cos 0 + c_2 sin 0) (c_3 cos cpt + $c_4$sin cpt)

0 = (c_1 +0) (c_3 cos cpt +c_4 sin cpt) [using the equation (2)]

0 = (c_1) (c_3 cos cpt +c_4 sin cpt)

[since t is a variable, therefore c_7 cos cpt +c_8 sin cpt $\neq$ 0]

0=c_1

Substitute the value of c_5 in the equation (6), we have

$y(x, t) = (0.\cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

$y(x, t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt)$ --------(7)

Apply boundary condition (3) in the equation (7), we have

i.e Substitute x=l in the equation (7), we have

$y(l, t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$

$0 = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$ [using the equation (3)]

$0 = c_2 \sin pl$ [since t is a variable, therefore $c_3 \cos cpt + c_4 \sin cpt \neq 0$]

$0 = \sin pl$ [If $c_2=0$, then $y(x, t) = 0$, therefore c_2 is nin zero]

$\sin n\pi = \sin pl$ [we know that, $\sin n\pi = 0$, for all n=1,2,3.....]

$n\pi = pl$ for all n=1,2,3.....

$\frac{n\pi}{l} = p$, for all n=1,2,3.....

Substitute the value of p in equation (7), we have

$y(x, t) = c_2 \sin \frac{n\pi}{l} x (c_3 \cos c\frac{n\pi}{l} t + c_4 \sin c\frac{n\pi}{l} t)$ for all n=1, 2, 3,.....

$y = y(x, t) = c_2 \sin \frac{n\pi}{l} x (c_3 \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t)$ for all n=1,2,3.......

---------(8)

Apply initial condition (4) in the equation (8), we have

Let us first differentiate equation (8) partially with respect to t, we have

$\frac{\partial}{\partial t} y(x, t) = c_2 \sin \frac{n\pi}{l} x (c_3 \frac{\partial}{\partial t} \cos \frac{cn\pi}{l} t + c_4 \frac{\partial}{\partial t} \sin \frac{cn\pi}{l} t)$

for all n=1,2,3...

$\frac{\partial}{\partial t} y(x, t) = c_2 \sin \frac{n\pi}{l} x (-c_3 \frac{l}{cn\pi} \sin \frac{cn\pi}{l} t + c_4 \frac{l}{cn\pi} \cos \frac{cn\pi}{l} t)$

for all n=1,2,3...(9)

Now substitute t=0 in the above equation (9), we have

$\frac{\partial}{\partial t}$y (x , t) $_{t=0}$ = c_2 sin $\frac{n\pi}{l}$x (-$c_3\frac{l}{cn\pi}$ sin 0 +c_4 $\frac{l}{cn\pi}$cos 0) for all n=1,2,3...

0= c_2 sin $\frac{n\pi}{l}$x (-$c_3\frac{l}{cn\pi}$ sin 0 +c_4 $\frac{l}{cn\pi}$cos 0) for all n=1,2,3...

[using the equation (iii)]

0= c_2 sin $\frac{n\pi}{l}$x (0 +c_4 $\frac{l}{cn\pi}$) for all n=1,2,3...

0= c_2 c_4 $\frac{l}{cn\pi}$ sin $\frac{n\pi}{l}$x for all n=1,2,3...

[since x is a variable, therefore sin px $\neq$ 0]

0= c_2 c_4 $\frac{l}{cn\pi}$ for all n=1,2,3...

[since $\frac{l}{cn\pi}$ $\neq$ 0, as l $\neq$ 0 and c $\neq$ 0]

0= c_2 c_4

0= c_4 [If c_2 =0, then u (x , t) becomes zero, therefore $c_2 \neq$ 0]

Substitute c_4 value in (8), we have

y(x , t) = c_2 sin $\frac{n\pi}{l}$x (c_3 cos $\frac{cn\pi}{l}$t + 0 . sin$\frac{cn\pi}{l}$t) for all n=1,2,3.......

y(x , t) = c_2 sin $\frac{n\pi}{l}$x (c_3 cos $\frac{cn\pi}{l}$t) for all n=1,2,3.......

y(x , t) = c_2 c_3 sin $\frac{n\pi}{l}$x cos $\frac{cn\pi}{l}$t for all n=1,2,3....... --------(10)

The general solution is given by

y(x , t) = $\sum_{n=1}^{\infty} b_n$ sin $\frac{n\pi}{l}$x cos $\frac{cn\pi}{l}$t where $b_n=c_2c_3$. -------(11)

Apply initial condition (5) in the equation (11), we have

i.e Substitute t=0 in equation (11), we have

y(x , 0) = $\sum_{n=1}^{\infty} b_n$ sin $\frac{n\pi}{l}$x

a sin $\frac{\pi x}{l}$ = $\sum_{n=1}^{\infty} b_n$ sin $\frac{n\pi}{l}$x

$a \sin \frac{\pi x}{l} + 0 + 0 \ldots = b_1 \sin \frac{\pi}{l} x + b_2 \sin \frac{2\pi}{l} x + \ldots + b_n \sin \frac{n\pi}{l} x + \ldots$

Equating the corresponding terms in the above equation, we have

$a = b_1$, and $b_n = 0$ for all $n = 2,3,4,\ldots$

Substitute the value of b_n for $n = 1,2,3\ldots$ in the equation (11), we have

$$y(x,t) = b_1 \sin \frac{\pi}{l} x \cos \frac{c\pi}{l} t + b_2 \sin \frac{2\pi}{l} x \cos \frac{2c\pi}{l} t + \ldots$$

$$y(x,t) = a \sin \frac{\pi}{l} x \cos \frac{c\pi}{l} t + 0 . \sin \frac{2\pi}{l} x \cos \frac{2c\pi}{l} t + \ldots$$

$$y(x,t) = a \sin \frac{\pi}{l} x \cos \frac{c\pi}{l} t$$

Which is the required solution.

Problem:-02

A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \left(\frac{\pi x}{l}\right)$. Find the displacement $y(x,t)$.

Solution:-

The one dimensional wave equation is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T}{m}$ --(1)

Boundary conditions are

$y(0,t) = 0$------(2)

$y(l,t) = 0$------(3)

Initial conditions are

$y(x,0) = 0$------(4)

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \left(\frac{\pi x}{l}\right)$ ------(5)

The solution of equation (1) is given by

$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ -------(6)

Apply boundary condition (2) in equation (6), we have

i.e Substitute x=0 in the equation (6), we have

$y(0,t) = (c_1 \cos 0 + c_2 \sin 0)(c_3 \cos cpt + c_4 \sin cpt)$

$0 = (c_1.1 + c_2.0)(c_3 \cos cpt + c_4 \sin cpt)$ [using the equation (2)]

$0 = c_1 (c_3 \cos cpt + c_4 \sin cpt)$

[since t is a variable, therefore $c_3 \cos cpt + c_4 \sin cpt \neq 0$]

$0 = c_1$

Substitute the value of c_1 in the equation (6), we have

$y(x,t) = (0.\cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$

$y(x,t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt)$ --------(7)

Apply boundary condition (3) in the equation (7), we have

i.e Substitute x=l in the equation (7), we have

$y(l,t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$

$0 = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt)$ [using the equation (3)]

$0 = c_2 \sin pl$

[since t is a variable, therefore $c_3 \cos cpt + c_4 \sin cpt \neq 0$]

$0 = \sin pl$ [If $c_2=0$, then $y(x,t) = 0$]

$\sin n\pi = \sin pl$

[we know that, $\sin n\pi = 0$, for all n=1,2,3.....]

$n\pi = pl$ for all n=1,2,3.....

$\frac{n\pi}{l} = p$, for all n=1,2,3.....

Substitute the value of p in equation (7), we have

$y(x,t) = c_2 \sin \frac{n\pi}{l} x \left(c_3 \cos c\frac{n\pi}{l} t + c_4 \sin c\frac{n\pi}{l} t\right)$ for all n=1, 2, 3,.....

$y(x,t) = c_2 \sin \frac{n\pi}{l} x \left(c_3 \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t\right)$ for all n=1,2,3.......

---------(8)

Apply initial condition (4) in the equation (8), we have

i.e Substitute t=0 in equation (8), we have

$y(x,0) = c_2 \sin \frac{n\pi}{l} x \left(c_3 \cos 0 + c_4 \sin 0\right)$ for all n=1,2,3.......

$0 = c_2 \sin \frac{n\pi}{l} x \left(c_3 .1 + c_4 .0\right)$ for all n=1,2,3......

. [using equation (4)]

$0 = c_2 c_3 \sin \frac{n\pi}{l} x$ for all n=1,2,3.......

$0 = c_2 c_3$ [since x is a variable, therefore $\sin \frac{n\pi}{l} x \neq 0$]

$0 = c_3$

[If $c_2=0$, then y (x , t) becomes zero, it should not be zero, therefore $c_2 \neq 0$]

Substitute the value of c_3 in equation (8), we have

$y(x,t) = c_2 \sin \frac{n\pi}{l} x \left(0 . \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t\right)$ for all n=1,2,3.......

$y(x,t) = c_2 \sin \frac{n\pi}{l} x \left(c_4 \sin \frac{cn\pi}{l} t\right)$ for all n=1,2,3.......

$y(x,t) = c_2 c_4 \sin \frac{n\pi}{l} x \sin \frac{cn\pi}{l} t$ for all n=1,2,3.......

The general solution is given by

$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \sin \frac{cn\pi}{l} t$ where $b_n = c_2 c_4$ ---------(9)

Apply initial condition (5) in the equation (9), we have

Let us first differentiate equation (9) partially with respect to t, we have

$$\frac{\partial}{\partial t} y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \frac{\partial}{\partial t} \sin \frac{cn\pi}{l} t$$

$$\frac{\partial}{\partial t} y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \left(\frac{cn\pi}{l}\right) \cos \frac{cn\pi}{l} t \qquad \text{---------(10)}$$

Substitute t=0 in the above equation (10), we have

$$\frac{\partial}{\partial t} y(x,t)\Big|_{t=0} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \left(\frac{cn\pi}{l}\right) \cos 0$$

$$v_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \left(\frac{cn\pi}{l}\right) \sin \frac{n\pi}{l} x \qquad \text{[using the equation (5)]}$$

$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$v_0 \frac{3\sin\frac{x\pi}{l} - \sin\frac{3x\pi}{l}}{4} = \sum_{n=1}^{\infty} b_n \left(\frac{cn\pi}{l}\right) \sin \frac{n\pi}{l} x$$

$$\frac{3v_0}{4}\sin\frac{x\pi}{l} - \frac{v_0}{4}\sin\frac{3x\pi}{l} = \left(\frac{c\pi}{l}\right) b_1 \sin \frac{\pi}{l} x + \left(\frac{2c\pi}{l}\right) b_2 \sin \frac{2\pi}{l} x$$

$$+ \left(\frac{3c\pi}{l}\right) b_3 \sin \frac{3\pi}{l} x + \ldots\ldots\ldots$$

Equating the corresponding terms on both sides of the above equation, we have

$\frac{3v_0}{4} = b_1 \frac{c\pi}{l}$ (equating the coefficient of $\sin\frac{x\pi}{l}$ on both sides)

$\Rightarrow \frac{3v_0 l}{4c\pi} = b_1$ and

$b_2=0$ (equating the coefficient of $\sin\frac{2x\pi}{l}$ on both sides)

$-\frac{v_0}{4} = b_3 \frac{3c\pi}{l}$ (equating the coefficient of $\sin\frac{3x\pi}{l}$ on both sides)

$\Rightarrow -\frac{l v_0}{12c\pi} = b_3$

$0 = b_4 = b_5 = \ldots\ldots$

Substitute all the b_i values in the equation (9), we have

$$y(x,t) = b_1 \sin \frac{\pi}{l} x \sin \frac{c\pi}{l} t + b_2 \sin \frac{2\pi}{l} x \sin \frac{2c\pi}{l} t$$

$$+ b_3 \sin \frac{3\pi}{l} x \sin \frac{3c\pi}{l} t \quad + b_4 \sin \frac{4\pi}{l} x \sin \frac{4c\pi}{l} t \ + \ldots\ldots$$

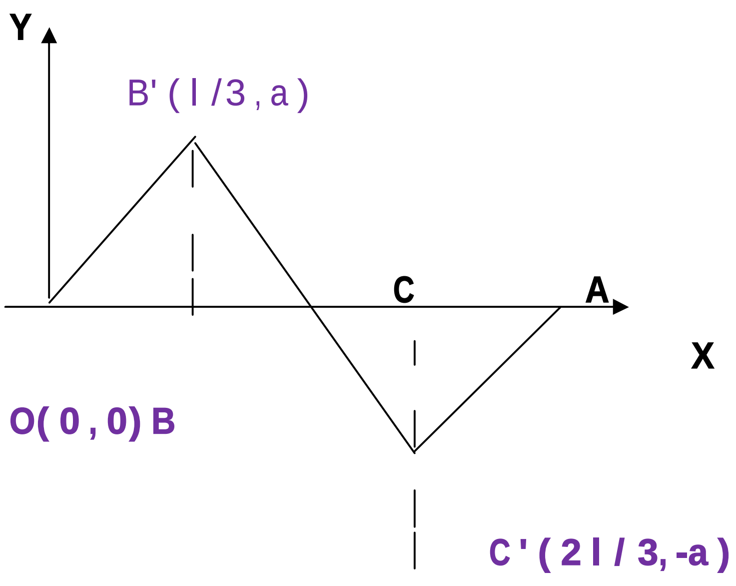
$$y(x,t) = \frac{3v_0 l}{4c\pi} \sin \frac{\pi}{l} x \sin \frac{c\pi}{l} t \ + 0.\ \sin \frac{2\pi}{l} x \sin \frac{2c\pi}{l} t$$

$$- \frac{l v_0}{12 c\pi} \sin \frac{3\pi}{l} x \sin \frac{3c\pi}{l} t \ + 0 .\sin \frac{4\pi}{l} x \sin \frac{4c\pi}{l} t \ + \ldots\ldots$$

$$y(x,t) = \frac{3v_0 l}{4c\pi} \sin \frac{\pi}{l} x \sin \frac{c\pi}{l} t \ - \frac{l v_0}{12 c\pi} \sin \frac{3\pi}{l} x \sin \frac{3c\pi}{l} t$$

Which is the required solution.

Note:-



Equation of the line OB'

It is a line joining two points O(0, 0) and B' (l/3, a)

Therefore it is given by $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$

$$\frac{x-0}{l/3-0}=\frac{y-0}{a-0}$$

=> $\frac{3x}{l}=\frac{y}{a}$

=> $\frac{3ax}{l}=y$ where 0 < x < l/3

Equation of the line B'C'

It is a line joining two points B' (l/3, a) and C' (2l/3, -a)

Therefore it is given by $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$

$$\frac{x-l/3}{2l/3-l/3}=\frac{y-a}{-a-a}$$

=> $\frac{x-l/3}{l/3}=\frac{y-a}{-2a}$

=> $\frac{3x-l}{l}=\frac{y-a}{-2a}$

=>- $\frac{2a(3x-l)}{l}+a=y$

=> $\frac{-2a(3x-l)+la}{l}=y$

=> $\frac{-6ax+2la+la}{l}=y$

=> $\frac{-6ax+3al}{l}=y$

=> $\frac{3a(l-2x)}{l}=y$ where l/3 < x < 2l/3.

Equation of the line C'A

It is a line joining two points C' (2l/3, -a) and A (l, 0)

Therefore it is given by $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$

$$\frac{x-2l/3}{l-2l/3}=\frac{y-(-a)}{0-(-a)}$$

=> $\frac{3x-2l}{3l-2l}=\frac{y+a}{a}$

=> $\dfrac{(3x-2l)a}{l} - a = y$

=>- $\dfrac{(3x-2l)a - la}{l} = y$

=> $\dfrac{3xa - 2la - la}{l} = y$

=> $\dfrac{3xa - 3la}{l} = y$

=> $\dfrac{3a(x-l)}{l} = y$

=> $\dfrac{3a(x-l)}{l} = y$ where 2l/3 < x < l.

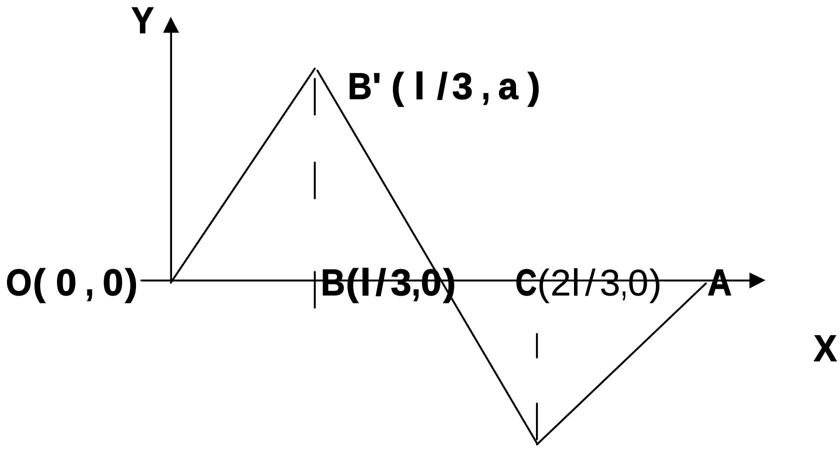
i.e y (x) = $\begin{cases} \dfrac{3a}{l}x & 0 \le x \le \dfrac{l}{3} \\ \dfrac{3a}{l}(l-2x) & \dfrac{l}{3} \le x \le \dfrac{2l}{3} \\ \dfrac{3a}{l}(x-l) & \dfrac{2l}{3} \le x \le l \end{cases}$

Problem:-03

A point of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid point of the string always remains at rest.

Solution:-

Let us draw the initial position of the string as follows



C ' (2 l / 3, -a)

Let B and C be the points of trisection of the string OA whose length is l.

Initial position of the string is OB'C'A, where BB'=CC'=a.

The one dimensional wave equation is $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ where $c^2 = \frac{T}{m}$ -(1)

Boundary conditions are

y (0 , t)= 0------(2)

y (l , t)= 0------(3)

Initial conditions are

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ -------(4)

$$y(x,0) = \begin{cases} \frac{3a}{l}x & 0 \le x \le \frac{l}{3} \\ \frac{3a}{l}(l-2x) & \frac{l}{3} \le x \le \frac{2l}{3} \\ \frac{3a}{l}(x-l) & \frac{2l}{3} \le x \le l \end{cases} \quad \text{------(5)}$$

The solution of equation (1) is given by

y (x , t) = ($c_1$cos px +c_2 sin px) (c_3 cos cpt +c_4 sin cpt) -------(6)

Apply boundary condition (2) in equation (6), we have

i.e Substitute x=0 in the equation (6), we have

y (0 , t) = ($c_1$cos 0 + c_2 sin 0) (c_3 cos cpt + $c_4$sin cpt)

0 = (c_1 +0) (c_3 cos cpt +c_4 sin cpt) [using the equation (2)]

0 = c_1 (c_3 cos cpt +c_4 sin cpt)

[since t is a variable, therefore c_7 cos cpt +c_8 sin cpt $\neq$ 0]

0=c_1

Substitute the value of c_1 in the equation (6), we have

y (x , t) = (0. cos px +c_2 sin px) (c_3 cos cpt +c_4 sin cpt)

$y(x,t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$ --------(7)

Apply boundary condition (3) in the equation (7), we have

i.e Substitute x=l in the equation (7), we have

$y(l,t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt)$

$0 = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt)$ [using the equation (3)]

$0 = c_2 \sin pl$

[since t is a variable, therefore $c_3 \cos cpt + c_4 \sin cpt \neq 0$]

$0 = \sin pl$ [If $c_2=0$, then $y(x,t) = 0$]

$\sin n\pi = \sin pl$ [we know that, $\sin n\pi = 0$, for all n=1,2,3.....]

$n\pi = pl$ for all n=1,2,3.....

$\frac{n\pi}{l} = p$, for all n=1,2,3.....

Substitute the value of p in equation (7), we have

$y(x,t) = c_2 \sin \frac{n\pi}{l}x\,(c_3 \cos c\frac{n\pi}{l}t + c_4 \sin c\frac{n\pi}{l}t)$ for all n=1, 2, 3,.....

$y(x,t) = c_2 \sin \frac{n\pi}{l}x\,(c_3 \cos \frac{cn\pi}{l}t + c_4 \sin \frac{cn\pi}{l}t)$ for all n=1,2,3...--(8)

Apply initial condition (4) in the equation (8), we have

Let us first differentiate equation (8) partially with respect to t, we have

$\frac{\partial}{\partial t}y(x,t) = c_2 \sin \frac{n\pi}{l}x\,(c_3 \frac{\partial}{\partial t}\cos \frac{cn\pi}{l}t + c_4 \frac{\partial}{\partial t}\sin \frac{cn\pi}{l}t)$

for all n=1,2,3...

$\frac{\partial}{\partial t}y(x,t) = c_2 \sin \frac{n\pi}{l}x\,(c_3\frac{cn\pi}{l}\ \sin \frac{cn\pi}{l}t + c_4 \frac{cn\pi}{l}\cos\frac{cn\pi}{l}t)$

for all n=1,2,3...(9)

Now substitute t=0 in the above equation (9), we have

$\frac{\partial}{\partial t}y(x,t)\ _{t=0} = c_2 \sin \frac{n\pi}{l}x\,(c_3\frac{cn\pi}{l}\ \sin 0 + c_4 \frac{cn\pi}{l}\cos 0)$ for all n=1,2,3...

$0 = c_2 \sin \frac{n\pi}{l}x\,(c_3\frac{cn\pi}{l}\ \sin 0 + c_4 \frac{cn\pi}{l}\cos 0)$ for all n=1,2,3...

[using the equation (iii)]

$0 = c_2 \sin \frac{n\pi}{l} x \left(0 + c_4 \frac{cn\pi}{l} \right)$ for all n=1,2,3...

$0 = c_2 c_4 \frac{cn\pi}{l} \sin \frac{n\pi}{l} x$ for all n=1,2,3...

$0 = c_2 c_4 \frac{cn\pi}{l}$ for all n=1,2,..[since x is a variable, therefore sin px $\neq$ 0]

$0 = c_2 c_4$ [since $\frac{cn\pi}{l} \neq 0$, as $l \neq 0$ and $c \neq 0$]

$0 = c_4$ [If $c_2=0$, then y (x , t) becomes zero, therefore $c_2 \neq 0$]

Substitute c_4 value in (8), we have

$y(x,t) = c_2 \sin \frac{n\pi}{l} x \left(c_3 \cos \frac{cn\pi}{l} t + 0 . \sin \frac{cn\pi}{l} t\right)$ for all n=1,2,3.......

$y(x,t) = c_2 \sin \frac{n\pi}{l} x \left(c_3 \cos \frac{cn\pi}{l} t\right)$ for all n=1,2,3.......

$y(x,t) = c_2 c_3 \sin \frac{n\pi}{l} x \cos \frac{cn\pi}{l} t$ for all n=1,2,3....... --------(10)

The general solution is given by

$y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos \frac{cn\pi}{l} t$ where $b_n = c_2 c_3$. ---------(11)

Apply initial condition (5) in the equation (11), we have

i.e Substitute t=0 in equation (11), we have

$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cos 0$ where $b_n = c_2 c_3$.

$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$ where $b_n = c_2 c_3$.

$$\Rightarrow \begin{cases} \frac{3a}{l} x & 0 \le x \le \frac{l}{3} \\ \frac{3a}{l}(l-2x) & \frac{l}{3} \le x \le \frac{2l}{3} \\ \frac{3a}{l}(x-l) & \frac{2l}{3} \le x \le l \end{cases} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$, $0 \le x \le l$

To find the value of b_n ,

We have to apply Fourier sine series over interval (0 , l).

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$b_n = \frac{2}{l}\left\{\int_0^{l/3} f(x)\sin\frac{n\pi x}{l}dx + \int_{l/3}^{2l/3} f(x)\sin\frac{n\pi x}{l}dx + \int_{2l/3}^{l} f(x)\sin\frac{n\pi x}{l}dx\right\}$$

$$b_n = \frac{2}{l}\left\{\int_0^{l/3} \frac{3a}{l}x\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3}\frac{3a}{l}(l-2x)\sin\frac{n\pi x}{l}dx + \int_{2l/3}^{l}\frac{3a}{l}(x-l)\sin\frac{n\pi x}{l}dx\right\}$$

$$b_n = \frac{6a}{l^2}\left\{\int_0^{l/3} x\sin\frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3}(l-2x)\sin\frac{n\pi x}{l}dx + \int_{2l/3}^{l}(x-l)\sin\frac{n\pi x}{l}dx\right\}$$

$$b_n = \frac{6a}{l^2}\left\{\begin{array}{l}\left[(x)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(-\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{l/3} + \left[(l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{l/3}^{2l/3} \\ +\left[(x-l)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{2l/3}^{l}\end{array}\right\}$$

$$b_n = \frac{6a}{l^2}\left\{\left[(l/3)\left(\frac{-\cos\frac{n\pi}{3}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi}{3}}{\frac{n^2\pi^2}{l^2}}\right)-(0)\left(\frac{-\cos 0}{\frac{n\pi}{l}}\right)+(1)\left(\frac{-\sin 0}{\frac{n^2\pi^2}{l^2}}\right)\right]+\right.$$

$$+\left[(l-4l/3)\left(\frac{-\cos\frac{2n\pi}{3}}{\frac{n\pi}{l}}\right)-(-2)\left(\frac{-\sin\frac{2n\pi}{3}}{\frac{n^2\pi^2}{l^2}}\right)-(l-2l/3)\left(\frac{-\cos\frac{n\pi}{3}}{\frac{n\pi}{l}}\right)+(-2)\left(\frac{-\sin\frac{n\pi}{3}}{\frac{n^2\pi^2}{l^2}}\right)\right]$$

$$\left.+\left[(l-l)\left(\frac{-\cos n\pi}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin n\pi}{\frac{n^2\pi^2}{l^2}}\right)-(2l/3-l)\left(\frac{-\cos\frac{2n\pi}{3}}{\frac{n\pi}{l}}\right)+(1)\left(\frac{-\sin\frac{2n\pi}{3}}{\frac{n^2\pi^2}{l^2}}\right)\right]\right\}$$

$$b_n = \frac{6a}{l^2}\left\{-\frac{l^2\cos\frac{n\pi}{3}}{3n\pi} + \frac{l^2\sin\frac{n\pi}{3}}{n^2\pi^2} + \frac{l^2\cos\frac{2n\pi}{3}}{3n\pi} - \frac{2l^2\sin\frac{2n\pi}{3}}{n^2\pi^2} + \frac{l^2\cos\frac{n\pi}{3}}{3n\pi} + \frac{2l^2\sin\frac{n\pi}{3}}{n^2\pi^2}\right.$$

$$\left. + \frac{l^2\sin n\pi}{n^2\pi^2} - \frac{l^2\cos n\pi\frac{2n\pi}{3}}{3n\pi} - \frac{l^2\sin\frac{2n\pi}{3}}{n^2\pi^2}\right\}$$

$$b_n = \frac{6a}{l^2}\left\{\frac{l^2\sin\frac{n\pi}{3}}{n^2\pi^2} - \frac{2l^2\sin\frac{2n\pi}{3}}{n^2\pi^2} + \frac{2l^2\sin\frac{n\pi}{3}}{n^2\pi^2} + \frac{l^2\sin n\pi}{n^2\pi^2} - \frac{l^2\sin\frac{2n\pi}{3}}{n^2\pi^2}\right\}$$

$$b_n = \frac{6al^2}{n^2\pi^2l^2}\left\{\sin\frac{n\pi}{3} + 2\sin\frac{2n\pi}{3} + 2\sin\frac{n\pi}{3} + \sin n\pi + \sin\frac{2n\pi}{3}\right\}$$

$$[\because \sin n\pi = 0 \text{ for all n} = 1,2,3....]$$

$$b_n = \frac{6a}{n^2\pi^2}\left\{3\sin\frac{n\pi}{3} - 3\sin\frac{2n\pi}{3}\right\}$$

$$b_n = \frac{18a}{n^2\pi^2}\left\{\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right\}$$

$$b_n = \frac{18a}{n^2\pi^2}\left\{\sin\frac{n\pi}{3} + (-1)^n\sin\frac{n\pi}{3}\right\} \qquad \left[\because \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n\sin\frac{n\pi}{3}\right]$$

$$b_n = \frac{18a}{n^2\pi^2}\left\{1 + (-1)^n\right\}\sin\frac{n\pi}{3}$$

Substitute the value of b_n in (11), we have

$$y(x,t) = \sum_{n=1}^{\infty}\frac{18a}{n^2\pi^2}\left\{1+(-1)^n\right\}\sin\frac{n\pi}{3}\sin\frac{n\pi}{l}x\cos\frac{cn\pi}{l}t$$

$$y(x,t) = \frac{18a}{\pi^2}\sum_{n=1}^{\infty}\frac{}{n^2}\left\{1+(-1)^n\right\}\sin\frac{n\pi}{3}\sin\frac{n\pi}{l}x\cos\frac{cn\pi}{l}t$$

Which is the required solution

Problem:-04

Derive the D' Alembert's Solution of Wave Equation

Solution:-

Consider the one dimensional wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$-----(1)

Let $D = \frac{\partial}{\partial t}$ & $D' = \frac{\partial}{\partial x}$

Hence equation (1) can be written as$(D^2-c^2D'^2)y=0$

y=y(x,t)=c.f+p.i

To c.f

The A.E is

$m^2-c^2=0$ (replace D by m and D' by 1)

m=+c,-c

The general solution of wave equation is C.F=f(x+ct)+g(x-ct)

P.I =0

y(x,t)=f(x+ct)+g(x-ct)+0=f(x+ct)+g(x-ct)-------(2)

where f and g are arbitrary function.

Suppose initially

y(x,0)= $\phi(x)$ -----(3)and

$\frac{\partial y(x,t)}{\partial t}\Big|_{t=0} = 0$ ----(4)

Apply (3) on (2)

sub t=0 in (2)

y(x,0)=f(x)+g(x)

$\phi(x)$ =f(x)+g(x) [using (3)

i.e f(x)+g(x)= $\phi(x)$ ------(5)

Apply (4) on (2), we get

Let us first differentiate(2) p.w.t 't'

$$\frac{\partial y(x,t)}{\partial t} = c\left(f'(x+ct) - g'(x-ct)\right)$$

substitute t=0 on both sides

$$\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = c\left(f'(x) - g'(x)\right)$$

$$0 = c\left(f'(x) - g'(x)\right) \quad \text{(using (4))}$$

$$f'(x) - g'(x) = 0$$

Integrating on both sides, we get

f(x)-g(x)=k-------(6)

From equation (5) & (6), we have

f(x)= $(\phi(x)+k)/2$ and g(x)= $(\phi(x)-k)/2$

Therefore equation (2) can be written as

y(x,t)= $(\phi(x+ct)+k)/2 + (\phi(x-ct)-k)/2$

This is the D'Alembert's solution of one dimensional wave equation.

EXERCISE

1. A tightly stretched string with fixed ends points x=0 and x=l is initially in a position given by y= $y_0 \sin^3\left(\frac{\pi x}{l}\right)$, If it is released from the rest from this position, find the displacement y(x,t). Hint. $\sin^3\left(\frac{\pi x}{l}\right)$ =

$$\frac{3\sin\frac{x\pi}{l} - \sin\frac{3x\pi}{l}}{4}$$

2. A string is stretched and fastened at two points $x = 0$ and $x = l$ apart. Motion is

started by displacing the string into the form $y = k(lx - x^2)$ from which it is released at time t=0. Find the displacement of any point on the string at a distance of x from one end at time t.

3. A tightly stretched string of length 2 l is fastened at both ends. The midpoint of the string is displaced by a distance `b' transversely and the string is released from rest in this position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

4. Find the displacement of any point of a string, if it is of length 2l and vibrating between fixed end points with initial velocity zero and initial displacement given by

$$f(x) = \begin{cases} \frac{kx}{l} & in \quad 0 < x < l \\ 2k - \frac{kx}{l} & \text{in } l < x < 2l. \end{cases}$$

5. A tightly stretched string with fixed end points x=0 and x=l is initially at rest in its equilibrium position. If it is set vibrating given each point a velocity $\lambda x(l-x)$ then show that y(x,t)=$\frac{8\lambda l^2}{\pi^4 a} \sum_{n=1,3,5.} \frac{1}{n^4} \sin\frac{n\pi x}{l} \sin\frac{na\pi t}{l}$.

6. A string of length l, at time t=0, the string is given a shape defined by f(x)=kx^2(l-x) where k is a constant and then released from rest. Find the displacement of any point of the string at any time t>0.

ONE - DIMENSIONAL HEAT FLOW

Consider a homogeneous bar of uniform cross-section α (cm^2).

Suppose that the sides are covered with a material impervious to heat so that the stream lines of heat flow are all parallel and perpendicular to the area α .Take one end of the bar as origin and the direction of flow ass the positive x-axis

Let p be the density , s be the specific heat and k the thermal conductivity

The one dimensional heat flow equation is given by

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad \text{where } a^2 = \frac{k}{s\rho} \text{ ----(1)}$$

Problem:-01

Derive the solution of one dimensional heat equation by the method of separation of variable.

Solution:-

W.k.t the one dimensional heat equation is $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$ ------(1)

Let u=X(x)T(t)------(2) be the solution of (1) , where X is a function x alone and T is a function of t alone.

Differentiate (2) partially w.r.t 't' , we get $\frac{\partial u}{\partial t} = XT'$ -------(3)

Differentiate (2) partially w.r.t 'x' twice, we get $\frac{\partial^2 u}{\partial x^2} = X''T$ -------(4)

Substitute (3) and (4) in (1), we get

XT'=a^{2}X''T

$\frac{T'}{a^2 T} = \frac{X''}{X} = k$ (say)------(5)

$\frac{T'}{a^2T}=k$ $T'=ka^2T$ $T'-ka^2T=0$------(6)	$\frac{X''}{X}=k$ $X''=kX$ $X''-kX=0$-----(7)

The equations (6) and (7) are ordinary differential equations, the solution of which depend on the value of k.

Case (i)

Let k=0

Equation (6) and (7) becomes

(6) => $T'-ka^2T=0$ $T'=0$ $dT/dt=0$ Integrate on both sides , $T=C_1$-----(8)	(7)=> $X''-kX=0$ $X''=0$ Integrate on both sides, $X'=C_2$ Integrate again, $X=C_2x+C_3$------(9)

Therefore $u(x,t)=C_1(C_2x+C_3)$

Case (ii)

Let k be positive, i.e $k=p^2$ (k is always positive irrespective of the value of p is +ve

or -ve)

(6)=> $T'-p^2 a^2T=0$ $\frac{dT}{dt}-p^2a^2T=0$ $\frac{dT}{T}=p^2a^2dt$ $\log T=p^2a^2t+\log C_4$ $T=e^{p^2a^2t+\log C_4}$ $T=C_4e^{p^2a^2t}$ --------(10)	(7)=> $X''-p^2 X=0$ The A.E is $m^2-p^2=0$ $m=p,-p$ $X=C_5e^{px}+C_6e^{-px}$--------(11)

Therefore $u(x,t)= C_4e^{p^2a^2t}$ ($C_5e^{px}+C_6e^{-px}$)

Case (III)

Let K be negative, i.e $k=-p^2$ (k is always negative irrespective of the value of p is +ve or -ve)

(6)=> $T'+p^2a^2T=0$ $\frac{dT}{dt}+p^2a^2T=0$ $\frac{dT}{T}=-p^2a^2dt$ $\log T=-p^2a^2t+\log C_7$ $T=e^{-p^2a^2t+\log C_7}$ $T=C_7e^{-p^2a^2t}$ ---------(12)	(7)=> $X''+p^2X=0$ The A.E is $m^2+p^2=0$ m=pi,-pi $X=C_8\cos px+C_9\sin px$---------(13)

Therefore $u(x,t)=C_7e^{-p^2a^2t}(C_8\cos px+C_9\sin px)$

From the above three cases, we have the following set of possible solutions for one dimensional heat flow equation.

(i) $u(x,t)=C_1(C_2x+C_3)$

(ii) $u(x,t)=C_4e^{p^2a^2t}(C_5e^{px}+C_6e^{-px})$

(iii) $u(x,t)=C_7e^{-p^2a^2t}(C_8\cos px+C_9\sin px)$

The solution (iii) is the only suitable solution of the heat equation.

Problem:-02

Solve the equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(x,0)=3\sin n\pi x$,

$u(0,t)=0$ and $u(1,t)=0$, where $0<x<1$, $t>0$.

Solution:-

The given one dimensional heat equation is $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ -(1)where $c^2=1$

The solution of the one dimensional heat equation (1) is

$u(x,t) = (c_1 \cos px + c_2 \sin px)e^{-c^2p^2t}$ -------(2)

Given boundary conditions are

u (0 , t) = 0 -------(3) and

u (1 , t) = 0 -------(4)

u (x, 0) = 3 sin nπ x --------(5) where 0 < x < 1, t > 0.

Apply the equation (3) in (2), we have

i.e Substitute x=0 in the equation (2), we have

$u(0,t) = (c_1 \cos 0 + c_2 \sin 0)e^{-c^2p^2t}$

$0 = (c_1.1 + c_2.0)e^{-c^2p^2t}$ [Using the equation (3)]

$0 = c_1.e^{-c^2p^2t}$ [since t is a variable, therefore $e^{-c^2p^2t} \neq 0$]

$0 = c_1$

Substitute the value of c_1 in equation (2), we have

$u(x,t) = (0.\cos px + c_2 \sin px)e^{-c^2p^2t}$

$u(x,t) = (c_2 \sin px)e^{-c^2p^2t}$

$u(x,t) = c_2 \sin px \; e^{-c^2p^2t}$ --------(6)

Apply the equation (4) in (6), we have

i.e Substitute x=1 in the equation (6), we have

$u(1,t) = c_2 \sin p \; e^{-c^2p^2t}$

$0 = c_2 \sin p \; e^{-c^2p^2t}$ [Using equation (4)]

$0 = c_2 \sin p$ [since t is a variable, therefore $e^{-c^2p^2t} \neq 0$]

$0 = \sin p$

[If c_2 =0 , then u(x,t) becomes zero, it should not be zero, therefore $c_2 \neq 0$]

sin nπ =sin p [we know that, sin nπ =0, for all n=1,2,3.....]

=> nπ =p, for n=1,2,3....

i.e p= nπ , for n=1,2,3....

Substitute the value of p in the equation (6), we have

$u(x,t) = c_2 \sin n\pi x \ e^{-c^2 n^2 \pi^2 t}$ for all n=1,2,3.... ---------(7)

The general solution is

$u(x,t) = b_n \sin n\pi x \ e^{-c^2 n^2 \pi^2 t}$ for n=1,2,3,...., where $b_n = c_2$ -------(8)

The complete solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin n\pi x \ e^{-c^2 n^2 \pi^2 t} \qquad \text{--------(9)}$$

Apply the equation (5) in (9), we have

i.e Substitute t=0 in the equation (9), we have

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin n\pi x \ e^{0}$$

$$3 \sin n\pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

$$3\sin n\pi x = b_1 \sin \pi x + b_2 \sin 2\pi x + \ldots.. + b_n \sin n\pi x$$
$$+ b_{n+1} \sin(n+1)\pi x + ..$$

$b_1=0$ (compare $\sin \pi x$)

$b_2=0$ (compare $\sin 2\pi x$)

;

$3=b_n$ (compare $\sin n\pi x$)

$0=b_{n+1}$ (compare $\sin(n+1)\pi x$)

.

Substitute the value of b_n in the equation (9), we have

$$u(x,t) = b_1 \sin \pi x \ e^{-c^2 1^2 \pi^2 t} + b_2 \sin 2\pi x \ e^{-c^2 2^2 \pi^2 t} + \ldots\ldots. + b_n \sin n\pi x \ e^{-c^2 n^2 \pi^2 t} +$$

$u(x,t) = 3\sin n\pi x \ e^{-c^2 n^2 \pi^2 t}$, where $c^2=1$

$u(x,t) = 3\sin n\pi x \ e^{-n^2 \pi^2 t}$,

Which is the required solution .

Problem:-03

Solve $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$ Subject to the boundary condition u(0,t)=0, u(l,t)=0 & u(x,0)=x.

Solution:-

The given one dimensional heat equation is $\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2}$ -------(1)

The solution of the one dimensional heat equation (1) is

$u(x,t)=(c_1\cos px+c_2\sin px)e^{-c^2p^2t}$ -------(2)

Given boundary conditions are

u (0 , t) = 0 -------(3) and

u (l , t) = 0 -------(4)

u (x, 0) = x --------(5)

Apply the equation (3) in (2), we have

i.e Substitute x=0 in the equation (2), we have

$u(0,t)=(c_1\cos 0+c_2\sin 0)e^{-c^2p^2t}$

$0=(c_1.1+c_2.0)e^{-c^2p^2t}$ [Using the equation (3)]

$0=c_1.e^{-c^2p^2t}$ [since t is a variable, therefore $e^{-c^2p^2t}\neq 0$]

$0=c_1$

Substitute the value of c_1 in equation (2), we have

$u(x,t)=(0.\cos px+c_2\sin px)e^{-c^2p^2t}$

$u(x,t)=(c_2\sin px)e^{-c^2p^2t}$

$u(x,t)=c_2\sin px\ \ e^{-c^2p^2t}$ --------(6)

Apply the equation (4) in (6), we have

i.e Substitute x=l in the equation (6), we have

$u(l,t) = c_2 \sin pl \;\; e^{-c^2 p^2 t}$

$0 = c_2 \sin pl \;\; e^{-c^2 p^2 t}$ [Using equation (4)]

$0 = c_2 \sin pl$ [since t is a variable, therefore $e^{-c^2 p^2 t} \neq 0$]

$0 = \sin pl$

[If $c_2 = 0$, then u(x,t) becomes zero, it should not be zero, therefore $c_2 \neq 0$]

sin nπ =sin pl [we know that, sin nπ =0, for all n=1,2,3.....]

=> nπ =p,

i.e p= $\frac{\pi n}{l}$, n=1,2,3.....

Substitute the value of p in the equation (6), we have

$u(x,t) = c_2 \sin n\pi x \;\; e^{-c^2 n^2 \pi^2 t}$ for all n=1,2,3.... ---------(7)

The general solution is

$u(x,t) = b_n \sin \frac{n\pi x}{l} \;\; e^{-c^2 n^2 \pi^2 t}$ for n=1,2,3,...., where $b_n = c_2$ --------(8)

The complete solution is

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \;\; e^{-c^2 n^2 \pi^2 t} \qquad \text{--------(9)}$$

Apply the equation (5) in (8), we have

i.e Substitute t=0 in the equation (9), we have

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \;\; e^0$$

$$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \text{------(*)}$$

Expand LHS of the above equation in a half range Fourier series over the interval (0,l).

$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l} dx = \frac{2}{l}\int_0^l x \sin \frac{n\pi x}{l} dx$

$$b_n=\frac{2}{l}\left\{(x)\left(-\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1).\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{x=0}^{x=l}$$

$$b_n=\frac{2}{l}\left\{\left(-\frac{lx\cos\frac{n\pi x}{l}}{n\pi}\right)-\left(-\frac{l^2\sin\frac{n\pi x}{l}}{n^2\pi^2}\right)\right\}_{x=0}^{x=l}$$

$$b_n=\frac{2}{l}\left\{\left(-\frac{lx\cos\frac{n\pi x}{l}}{n\pi}\right)+\left(\frac{l^2\sin\frac{n\pi x}{l}}{n^2\pi^2}\right)\right\}_{x=0}^{x=l}$$

$$b_n=\frac{2}{l}\left\{\left(-\frac{l^2\cos n\pi}{n\pi}\right)+\left(\frac{l^2\sin n\pi}{n^2\pi^2}\right)-\left(-\frac{l^2.0.\cos 0}{n\pi}\right)-\left(\frac{l^2\sin 0}{n^2\pi^2}\right)\right\}$$

$$b_n=\frac{2}{l}\left\{-\left(\frac{l^2\cos n\pi}{n\ \pi}\right)\right\}$$

$\sin n\ \pi=0$ for all n=1,2,3....

$$b_n=\frac{-2l\cos n\ \pi}{n\ \pi}$$

$\cos n\ \pi=(-1)^n$ for all n=1,2,3....

$$b_n=\frac{-2l(-1)^n}{n\ \pi}$$

$$b_n=\frac{2l(-1)^{n+1}}{n\ \pi}$$ where n=1,2,3......

Substitute in (9), we have

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2l(-1)^{n+1}}{n\ \pi}\sin\frac{n\pi x}{l}\ e^{-c^2n^2\pi^2t}$$

Which is the required solution .

Problem:-04

Find the solution to the equation $\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2}$ that satisfies the conditions (i) u(0,t)=0 (ii) u(l,t)=0 for t>0 and (iii) u(x,0)= $\begin{cases} x & 0<x<l/2 \\ l-x & l/2<x<l \end{cases}$

Solution:-

The one dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

The boundary conditions are

(i) u(0,t)=0 for t>0

(ii) u(l,t)=0 for t>0

(iii) u(x,0)= $\begin{cases} x & 0 < x < l/2 \\ l - x & l/2 < x < l \end{cases}$

The solution of one dimensional heat equation is

u(x,t)=(Acos px +B sin px) $e^{-\alpha^2 p^2 t}$---------(1)

Apply condition (i) on (1), we get

Sub t=0 in (1), we get

u(0,t)=(Acos 0 +B sin 0) $e^{-\alpha^2 p^2 t}$

0=(A) $e^{-\alpha^2 p^2 t}$

0=A [$e^{-\alpha^2 p^2 t}$ is non zero]

Sub the value of A in (1), we get

u(x,t)=B sin px $e^{-\alpha^2 p^2 t}$---------(2)

Apply condition (ii) on (2), we get

Sub x=l in (2), we get

u(l,t)=B sin pl $e^{-\alpha^2 p^2 t}$

0=B sin pl $e^{-\alpha^2 p^2 t}$

0=B sin pl [$e^{-\alpha^2 p^2 t}$ is non zero]

0=sin pl [B cannot be zero]

sin πn=sin pl, for n=1,2,3,....

pl=πn, for n=1,2,3...

p=$\frac{\pi n}{l}$ for n=1,2,3...

Sub the value of p in (2) , we get

u(x,t)=B sin $\frac{\pi nx}{l}$ $e^{-\frac{\alpha^2\pi^2n^2t}{l^2}}$ ---------(3) for n=1,2,3...

The most general solution is

u(x,t)= $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi nx}{l}$ $e^{-\frac{\alpha^2\pi^2n^2t}{l^2}}$ ---------(4)

Applying condition (iii) on (4) ,we get

Sub t=0 in (4), we get

u(x,0)= $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi nx}{l}$ e^0

u(x,0)=f(x)= $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi nx}{l}$, where u(x,0)= $\begin{cases} x & 0 < x < l/2 \\ l - x & l/2 < x < l \end{cases}$

To find B_n

Use half range Fourier sine series over the interval 0<x<l, we get

$$B_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$B_n=\frac{2}{l}\int_0^{l/2} f(x)\sin\frac{n\pi x}{l}dx+\frac{2}{l}\int_{l/2}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

$$B_n=\frac{2}{l}\int_0^{l/2} x\sin\frac{n\pi x}{l}dx+\frac{2}{l}\int_{l/2}^{l} (l-x)\sin\frac{n\pi x}{l}dx$$

$$B_n=\frac{2}{l}\left\{x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-1.\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^{l/2}+\frac{2}{l}\left\{(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(-1).\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_{l/2}^{l}$$

$$B_n=\frac{2}{l}\left\{(l/2)\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{l}}\right)-1.\left(\frac{-\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right)-(0)\left(\frac{-\cos 0}{\frac{n\pi}{l}}\right)+1.\left(\frac{-\sin 0}{\frac{n^2\pi^2}{l^2}}\right)\right\}$$

$$+\frac{2}{l}\left\{(l-l)\left(\frac{-\cos n\pi}{\frac{n\pi}{l}}\right)-(-1).\left(\frac{-\sin n\pi}{\frac{n^2\pi^2}{l^2}}\right)-(l-l/2)\left(\frac{-\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}\right)+(-1).\left(\frac{-\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right)\right\}$$

$$B_n = \frac{2}{l}\left\{\left(\frac{-l^2\cos\frac{n\pi x}{2}}{2n\pi}\right)+.\left(\frac{l^2\sin\frac{n\pi}{2}}{n^2\pi^2}\right)\right\}+\frac{2}{l}\left\{(l/2)\left(\frac{\cos\frac{n\pi}{2}}{\frac{n\pi}{l}}\right)+.\left(\frac{\sin\frac{n\pi}{2}}{\frac{n^2\pi^2}{l^2}}\right)\right\}$$

$$B_n = \frac{2}{l}\left\{\frac{-l^2\cos\frac{n\pi x}{2}}{2n\pi}+\frac{l^2\sin\frac{n\pi}{2}}{n^2\pi^2}\right\}+\frac{2}{l}\left\{\frac{l^2\cos\frac{n\pi}{2}}{2n\pi}+.\frac{l^2\sin\frac{n\pi}{2}}{n^2\pi^2}\right\}$$

$$B_n = \frac{2}{l}\left\{\frac{l^2\sin\frac{n\pi}{2}}{n^2\pi^2}+\frac{l^2\sin\frac{n\pi}{2}}{n^2\pi^2}\right\}$$

$$B_n = \frac{4l\ \sin\frac{n\pi}{2}}{n^2\pi^2}$$

Sub B_n in (4), we get

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4l\ \sin\frac{n\pi}{2}}{n^2\pi^2}\sin\frac{\pi n x}{l}\ e^{-\frac{\alpha^2\pi^2 n^2 t}{l^2}}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4l}{n^2\pi^2}\ \sin\frac{n\pi}{2}\sin\frac{\pi n x}{l}\ e^{-\frac{\alpha^2\pi^2 n^2 t}{l^2}}$$

Which is the required solution.

STEADY STATE CONDITIONS

Steady state condition in heat flow means that the temperature at any point in the body does not vary with time. i.e it is independent of time t.

Problem:-01

Derive the solution of one dimensional heat flow equation under steady state.

Solution:-

The one dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ ---------(1)

Let us assume the steady state conditions prevails.

In steady state condition, the temperature(u) is depend only on x and not on t.

Hence $\frac{\partial u}{\partial t} = 0$-----(2)

Sub (2) in (1), we get $\frac{\partial^2 u}{\partial x^2} = 0$-------(3)

Integrate (3) w.r.t x, we get

$\frac{\partial u}{\partial x} = a$

Integrate again w.r.t x, we get

u(x)=ax+b------(4),where a and b are arbitrary constant.

Which is the required general solution.

Problem:-02(steady state and zero boundary conditions)

A rod of 30cm long has its ends A and B kept at 20^o and 80^o respectively until steady state conditions prevails. The temperature at each end is then suddenly reduced to 0^oc, and kept so. Find the resulting temperature u(x,t) taking x=0 at A.

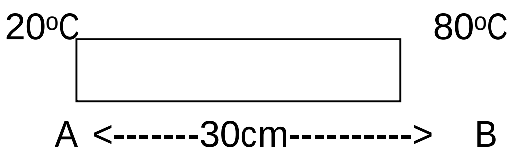
Solution:-

The temperature function u(x,t) is the solution of the one dimensional heat equation

$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$-------(1)

When the steady state condition prevails

In steady state condition, the temperature is depend only on x and not on t.



Hence $\frac{\partial u}{\partial t} = 0$

Sub (1), we get $\frac{\partial^2 u}{\partial x^2} = 0$

Integrate w.r.t x, we get

$$\frac{\partial u}{\partial x} = a$$

Integrate again w.r.t x, we get

u(x)=ax+b------(2), where a and b are arbitrary constant.

At the end A (x=0)

Since the temperature is 20^o

u (0)=20----(*),

Apply(*) on (2), we have

sub x=0 in (2)

u(0)=a.0+b

20=b [use (3)]

Sub the value of b in (2), we have

u(x)=ax+20------(3)

At the end B (x=30)

Since the temperature is 80^o

u(30)=80---(**)

Apply(**)on (3), we have

sub x=30 in (3)

u(30)=30a+20

80=30a+20

a=2

Sub the value of a in (3), we have

u(x)=2x+20------(4)

Hence the boundary and initial conditions are

(i) u(0,t)=0 for all t>0 (at the end A)

(ii) u(30,t)=0 for all t>0 (at the end B)

(iii) u(x,0)=2x+20

Now the suitable solution which satisfies our boundary conditions is given by u(x,t)=(Acos px +B sin px) $e^{-\alpha^2 p^2 t}$ -------(5)

Apply condition (i) on (5), we get

Sub x=0 in (5)

u(0,t)=(Acos 0 +B sin 0) $e^{-\alpha^2 p^2 t}$

0=(A) [$e^{-\alpha^2 p^2 t}$ is non zero]

Sub the value of A in (5), we get

u(x,t)=(B sin px) $e^{-\alpha^2 p^2 t}$ -------(6)

Apply condition (ii) on (6) , we get

Sub x=30, we get

u(30,t)=(B sin 30p) $e^{-\alpha^2 p^2 t}$

0=(B sin 30p) [$e^{-\alpha^2 p^2 t}$ is non zero]

0=sin 30p [B cannot be zero]

sin πn =sin 30p, for n=1,2,3,....

30p= πn , for n=1,2,3...

p= $\frac{\pi n}{30}$ for n=1,2,3...

Sub the value of p in (6) , we get

u(x,t)=B sin $\frac{\pi n x}{30}$ $e^{-\frac{\alpha^2 \pi^2 n^2 t}{30^2}}$ ---------(7) for n=1,2,3...

The most general solution is

u(x,t)= $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi n x}{30}$ $e^{-\frac{\alpha^2 \pi^2 n^2 t}{30^2}}$ ---------(8)

Applying condition (iii) on (8) ,we get

Sub t=0 in (8), we get

u(x,0)= $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi n x}{30}$ e^0

f(x)=2x+20= $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi n x}{30}$

To find B_n

Use half range Fourier sine series over the interval 0<x<30, we get

$$B_n=\frac{2}{l}\int_0^l u(x,0)\sin\frac{n\pi x}{l}dx$$

$$B_n=\frac{2}{30}\int_0^{30} u(x,0)\sin\frac{n\pi x}{30}dx$$

$$B_n=\frac{1}{15}\int_0^{30} (2x+20)\sin\frac{n\pi x}{30}dx$$

$$B_n=\frac{1}{15}\left\{(2x+20)\left(\frac{-\cos\frac{n\pi x}{30}}{\frac{n\pi}{30}}\right)-(2)\left(\frac{-\sin\frac{n\pi x}{30}}{\frac{n^2\pi^2}{30^2}}\right)\right\}_0^{30}$$

$$B_n=\frac{1}{15}\left\{\left(\frac{-30(2x+20)\cos\frac{n\pi x}{30}}{n\pi}\right)+\left(\frac{2.30^2\sin\frac{n\pi x}{30}}{n^2\pi^2}\right)\right\}_0^{30}$$

$$B_n=\frac{1}{15}\left\{\left(\frac{-30(60+20)\cos n\pi}{n\pi}\right)+\left(\frac{2.30^2\sin n\pi}{n^2\pi^2}\right)-\left(\frac{-30(20)\cos 0}{n\pi}\right)-\left(\frac{2.30^2\sin 0}{n^2\pi^2}\right)\right\}$$

$$B_n=\frac{1}{15}\left\{\left(\frac{-2400\cos n\pi}{n\pi}\right)+\left(\frac{600}{n\pi}\right)\right\}$$

$$B_n=\frac{600}{15n\pi}\left\{-4(-1)^n+1\right\}$$

$$B_n=\frac{40}{n\pi}\left\{1-4(-1)^n\right\}$$

Sub B_n value in (8), we get

$$u(x,t)=\sum_{n=1}^{\infty}\frac{40}{n\pi}\left\{1-4(-1)^n\right\}\sin\frac{\pi nx}{30}\,e^{-\frac{\alpha^2\pi^2n^2t}{30^2}}$$

Which is the required solution.

Problem-02(steady state and zero boundary conditions)

An insulated end of length l has its ends A and B kept at a° and b° Celsius respectively until steady state conditions prevails. The temperature at each end is suddenly reduced to zero degree Celsius and kept so. Find the resulting temperature at any point of the rod taking the end A as origin.

Solution:-

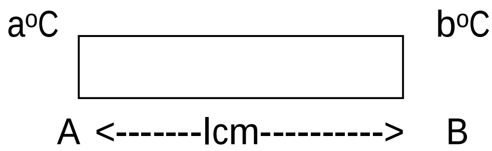
The temperature function u(x,t) is the solution of the one dimensional heat equation

$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ -------(1)

When the steady state condition prevails

In steady state condition, the temperature is depend only on x and not on t.

Hence $\frac{\partial u}{\partial t} = 0$



Sub (1), we get $\frac{\partial^2 u}{\partial x^2} = 0$

Integrate w.r.t x, we get

$\frac{\partial u}{\partial x} = a$

Integrate again w.r.t x, we get

u(x)=ax+b------(2), where a and b are arbitrary constant.

At the end A

x=0 and u (0)=a^o, apply this condition on (2), we have

u(0)=a.0+b

a^o=b

Sub the value of b in (2), we have

u(x)=ax+a^o------(3)

At the end B

x=l and u(l)=b^o, apply this condition on (3), we have

u(l)=la+a^o

b^o=la+a^o

a=(b^o-a^o)/l

Sub the value of a in (3), we have

u(x)=(b^o-a^o)x/l +a^o------(4)

Hence the boundary and initial conditions are

(i) u(0,t)=0 for all t>0

(ii) u(l,t)=0 for all t>0

(iii) u(x,0)= (b^o-a^o)x/l +a^o

Now the suitable solution which satisfies our boundary conditions is given by

u(x,t)=(Acos px +B sin px) $e^{-\alpha^2 p^2 t}$--------(5)

Apply condition (i) on (5), we get

Sub x=0 in (5)

u(0,t)=(Acos 0 +B sin 0) $e^{-\alpha^2 p^2 t}$

0=(A) [$e^{-\alpha^2 p^2 t}$ is non zero]

Sub the value of A in (5), we get

u(x,t)=(B sin px) $e^{-\alpha^2 p^2 t}$--------(6)

Apply condition (ii) on (6) , we get

Sub x=l, we get

u(l,t)=(B sin lp) $e^{-\alpha^2 p^2 t}$

0=(B sin lp) [$e^{-\alpha^2 p^2 t}$ is non zero]

0=sin lp [B cannot be zero]

sin πn=sin lp, for n=1,2,3,....

lp=πn, for n=1,2,3...

p=$\frac{\pi n}{l}$ for n=1,2,3...

Sub the value of p in (6) , we get

u(x,t)=B sin $\frac{\pi n x}{l}$ $e^{-\frac{\alpha^2 \pi^2 n^2 t}{l^2}}$ ---------(7) for n=1,2,3...

The most general solution is

$$u(x,t)= \sum_{n=1}^{\infty} B_n \sin \frac{\pi n x}{l} \; e^{-\frac{\alpha^2\pi^2 n^2 t}{l^2}} \text{ ---------(8)}$$

Applying condition (iii) on (8) ,we get

Sub t=0 in (8), we get

$$u(x,0)= \sum_{n=1}^{\infty} B_n \sin \frac{\pi n x}{l} \; e^0$$

$$u(x,0)= \sum_{n=1}^{\infty} B_n \sin \frac{\pi n x}{l}$$, where $u(x,0)=(b^o-a^o)x/l+a^o$

To find B_n

Use half range Fourier sine series over the interval 0<x<30, we get

$$B_n=\frac{2}{l}\int_0^l u(x,0)\sin\frac{n\pi x}{l}dx$$

$$B_n=\frac{2}{l}\int_0^l u(x,0)\sin\frac{n\pi x}{l}dx$$

$$B_n=\frac{2}{l}\int_0^l [(b^o - a^o)x/l + a^o]\sin\frac{n\pi x}{l}dx$$

$$B_n=\frac{2}{l}\left\{[(b^o-a^o)x/l+a^o]\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-[(b^o-a^o)/l]\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^l$$

$$B_n=\frac{2}{l}\left\{\left(\frac{-l[(b^o-a^o)x/l+a^o]\cos\frac{n\pi x}{l}}{n\pi}\right)+\left(\frac{[(b^o-a^o)/l].l^2\sin\frac{n\pi x}{l}}{n^2\pi^2}\right)\right\}_0^l$$

$B_n=$

$$\frac{2}{l}\left\{\left(\frac{-l[(b^o-a^o)+a^o]\cos n\pi}{n\pi}\right)+\left(\frac{[(b^o-a^o)/l].l^2\sin n\pi}{n^2\pi^2}\right)-\left(\frac{-l[a^o]\cos 0}{n\pi}\right)+\left(\frac{[(b^o-a^o)/l].l^2\sin 0}{n^2\pi^2}\right)\right\}$$

$$B_n=\frac{2}{l}\left\{\left(\frac{-l[(b^o-a^o)+a^o]\cos n\pi}{n\pi}\right)+-\left(\frac{-l[a^o]\cos 0}{n\pi}\right)\right\}$$

$$B_n=\frac{2}{n\pi}\left\{-[(b^o-a^o)+a^o]\cos n\pi + a^o\cos 0\right\}$$

$B_n = \frac{2}{n\pi}\left\{-b^o(-1)^n + a^o\right\}$

Sub B_n value in (8), we get

$u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left\{-b^o(-1)^n + a^o\right\}\sin\frac{\pi nx}{l}\ e^{-\frac{\alpha^2\pi^2n^2t}{l^2}}$

Which is the required solution.

Problem:-01(Steady state conditions and Non-Zero boundary conditions)

The end A and B of a rod 30cm long have their temperature kept at 20^o and the another end at 80^o until the steady state condition prevails. The temperature of the end B is suddenly reduced to 60^o and kept to while the end A is raised to 40^o. Find the temperature distribution in the rod after time t.

Solution:-

The temperature function u(x,t) is the solution of the one dimensional heat equation

$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ --------(1)

The solution may be $u(x,t) = u_s(x) + u_t(x,t)$------(2)

Steady state condition-1	Steady state condition-2
A B x=0 x=30 u=20 u=80 (i) u(0)=20 (ii) u(30)=80 u(x)=ax+b apply (i), we get u(0)=b 20=b	A B x=0 x=30 u=40 u=60 (i) u(0)=40 (ii) u(30)=60 u(x)=ax+b apply (i), we get u(0)=b

Sub the value of b, we get u=ax+20 Aplly (ii), we get U(30)=30a+20 80=30a+20 a=2 Sub the value of a, we get u(x)=2x+20	40=b Sub the value of b, we get u=ax+40 Aplly (ii), we get U(30)=30a+40 60=30a+40 a=2/3 Sub the value of a, we get u_s(x)=2x/3+40

Substitute the above values in (2), we get

u(x,t)=2x/3+40+u_t(x,t)---------(3)

The boundary conditions are

(a) u(0,t)=40, for t>0

(b) u(30,t)=60, for t>0

(c) u(x,0)=2x+20

Now the suitable solution which satisfies our boundary conditions is given by

u(x,t)=2x/3+40+(Acos px +B sin px) $e^{-\alpha^2 p^2 t}$-------(4)

Apply condition (i) on (4), we get

Sub x=0 in (4)

u(0,t)=40+(Acos 0 +B sin 0) $e^{-\alpha^2 p^2 t}$

40=(A+40) [$e^{-\alpha^2 p^2 t}$ is non zero]

A=0

Sub the value of A in (4), we get

u(x,t)=2x/3+40+(B sin px) $e^{-\alpha^2 p^2 t}$-------(5)

Apply condition (ii) on (5) , we get

Sub x=30, we get

u(30,t)=20+40+(B sin 30p) $e^{-\alpha^2 p^2 t}$

60=60+(B sin 30p) [$e^{-\alpha^2 p^2 t}$ is non zero]

0=sin 30p [B cannot be zero]

sin πn=sin 30p, for n=1,2,3,....

30p=πn, for n=1,2,3...

p=$\frac{\pi n}{30}$ for n=1,2,3...

Sub the value of p in (5) , we get

u(x,t)=2x/3+40+B sin $\frac{\pi nx}{30}$ $e^{-\frac{\alpha^2\pi^2 n^2 t}{30^2}}$ ---------(6) for n=1,2,3...

The most general solution is

u(x,t)=2x/3+40+ $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi nx}{30}$ $e^{-\frac{\alpha^2\pi^2 n^2 t}{30^2}}$ ---------(7)

Applying condition (iii) on (7) ,we get

Sub t=0 in (7), we get

u(x,0)= 2x/3+40+$\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi nx}{30}$ e^0

u(x,0)=2x/3+40+ $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi nx}{30}$, where u(x,0)=2x+20

2x+20=2x/3+40+ $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi nx}{30}$

2x+20-2x/3-40= $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi nx}{30}$

4x/3-20= $\sum_{n=1}^{\infty} B_n$ sin $\frac{\pi nx}{30}$

To find B_n

Use half range Fourier sine series over the interval 0<x<30, we get

B_n=$\frac{2}{30}\int_0^{30} u(x,0)\sin\frac{n\pi x}{30}dx$

B_n=$\frac{1}{15}\int_0^{30} (4x/3-20)\sin\frac{n\pi x}{30}dx$

$$B_n=\frac{1}{15}\left\{[4x/3-20]\left(\frac{-\cos\frac{n\pi x}{30}}{\frac{n\pi}{30}}\right)-[4/3]\left(\frac{-\sin\frac{n\pi x}{30}}{\frac{n^2\pi^2}{30^2}}\right)\right\}_0^{30}$$

$$B_n=\frac{1}{15}\left\{[40-20]\left(\frac{-\cos n\pi}{\frac{n\pi}{30}}\right)-[4/3]\left(\frac{-\sin n\pi}{\frac{n^2\pi^2}{30^2}}\right)-[-20]\left(\frac{-\cos 0}{\frac{n\pi}{30}}\right)+[4/3]\left(\frac{-\sin 0}{\frac{n^2\pi^2}{30^2}}\right)\right\}$$

$$B_n=\frac{1}{15}\left\{\left(\frac{-600\cos n\pi}{n\pi}\right)-\left(\frac{600\cos 0}{n\pi}\right)\right\}$$

$$B_n=\frac{-40}{n\pi}\left\{(-1)^n+1\right\}$$

Sub B_n value in (7), we get

$$u(x,t)=2x/3+40+\sum_{n=1}^{\infty}\frac{-40}{n\pi}\left\{(-1)^n+1\right\}\sin\frac{\pi nx}{30}\,e^{-\frac{\alpha^2\pi^2n^2t}{30^2}}$$

$$u(x,t)=2x/3+40-\frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{(-1)^n+1\right\}\sin\frac{\pi nx}{30}\,e^{-\frac{\alpha^2\pi^2n^2t}{30^2}}$$

Which is the required solution.

TEMPERATURE GRADIENT

The rate of change of temperature with respect to distance is called temperature gradient and it denoted by $\frac{\partial u}{\partial x}$

FOURIER LAW OF HEAT CONDUCTION

The rate at which heat flows across an area A at a distance x from one end of a bar is given by Q=-KA($\frac{\partial u}{\partial x}$)$_x$, where K= thermal conductivity, and ($\frac{\partial u}{\partial x}$)$_x$ is temperature gradient at x.

Thermally insulated ends

If there will be no heat flow passes through the ends of the bar then that two ends said to be thermally insulated.

By Fourier law we have Q=0 at both ends.

i.e -KA($\frac{\partial u}{\partial x}$)$_x$=0 at both ends

i.e ($\frac{\partial u}{\partial x}$)$_x$=0 at both ends

i.e ($\frac{\partial u}{\partial x}$)$_{at\ x=0}$=0 and ($\frac{\partial u}{\partial x}$)$_{at\ x=l}$=0

One end is thermally insulated

Problem:-01

Solve the problem of heat conduction in a rod given that (i) $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (ii) u is finite as t tends to infinity (iii) $\frac{\partial u}{\partial x} = 0$ for x=0 & x=l, t>0 (iv) u=lx-x^2 for t=0, $0 \le x \le l$

Solution

The one dimensional heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$-----(1)

On solving this equation (1) by the method of separation of variable and applying condition (ii) , we get the correct solution of the form

u(x,t)=(Acos px +B sin px) $e^{-\alpha^2 p^2 t}$-------(2)

Now condition (iii) can be rewritten as follows

$\frac{\partial u(0,t)}{\partial x} = 0$-----(a)

$\frac{\partial u(l,t)}{\partial x} = 0$ ------(b)

Differentiating (2) partially w.r.t x, we get

$\frac{\partial u(x,t)}{\partial x}$ =(-Ap sin px +Bp cos px) $e^{-\alpha^2 p^2 t}$ --------(3)

Now applying condition (a) on (3), we get

sub x=0 in (3)

$\frac{\partial u(0,t)}{\partial x}$ =(-Ap sin 0 +Bp cos 0) $e^{-\alpha^2 p^2 t}$

0=Bp $e^{-\alpha^2 p^2 t}$

0=Bp [$e^{-\alpha^2 p^2 t}$ in non zero]

0=B [p cannot be zero]

Sub the value of B in (2) , we get

u(x,t)=(Acos px) $e^{-\alpha^2 p^2 t}$ -------(4)

Now applying condition (b) on (4), we get

Differentiate (4) partially w.r.t x, we get

$\frac{\partial u(x,t)}{\partial x}$ =-Ap sin px $e^{-\alpha^2 p^2 t}$ -------(5)

Sub x=l in (5), we get

$\frac{\partial u(l,t)}{\partial x}$ =-Ap sin pl $e^{-\alpha^2 p^2 t}$

0=-Ap sin pl $e^{-\alpha^2 p^2 t}$

0=-Ap sin pl [$e^{-\alpha^2 p^2 t}$ is non zero]

0= sin pl [A , p are cannot be zero]

sin πn =sin pl, for n=0, 1,2,3,....

pl= πn , for n=0, 1,2,3...

p= $\frac{\pi n}{l}$ for n=0, 1,2,3...

Sub the value of p in (4) , we get

u(x,t)=A $\cos\frac{\pi nx}{l}\; e^{-\frac{\alpha^2\pi^2n^2t}{l^2}}$ ---------(6) for n=0,1,2,3...

The most general solution is

u(x,t)=$A_0+\sum_{n=1}^{\infty}A_n\cos\frac{\pi nx}{l}\; e^{-\frac{\alpha^2\pi^2n^2t}{l^2}}$ ---------(7)

Applying (iv) on (7), we get

Sub t=0 in (7), we get

u(x,0)= $A_0+\sum_{n=1}^{\infty}A_n\cos\frac{\pi nx}{l}\; e^{0}$

$lx-x^2= A_0+\sum_{n=1}^{\infty}A_n\cos\frac{\pi nx}{l}$ -------(8)

To find A_0 & An

Use half rand cosine series for the function $lx-x^2$ over the interval $0<x<l$.

$lx-x^2=a_0/2+\sum_{n=1}^{\infty}a_n\cos\frac{\pi nx}{l}$ -------(9)

$a_0=\frac{2}{l}\int_0^l u(x,0)dx$

$a_0=\frac{2}{l}\int_0^l (lx-x^2)dx$

$a_0=\frac{2}{l}\left(l\frac{x^2}{2}-\frac{x^3}{3}\right)_0^l$

$a_0=\frac{2}{l}\left(l\frac{l^2}{2}-\frac{l^3}{3}\right)$

$a_0=\frac{2l^3}{l}\left(\frac{1}{2}-\frac{1}{3}\right)$

$a_0=\frac{2l^2}{6}=\frac{l^2}{3}$

$a_n=\frac{2}{l}\int_0^l u(x,0)\cos\frac{n\pi x}{l}dx$

$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$

$a_n = \frac{2}{l}\left\{(lx - x^2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right\}_0^l$

$a_n = \frac{2}{l}\left\{(l^2 - l^2)\left(\frac{-\sin n\pi}{\frac{n\pi}{l}}\right) - (l-2l)\left(\frac{-\cos n\pi}{\frac{n^2\pi^2}{l^2}}\right) - (0-0)\left(\frac{-\sin 0}{\frac{n\pi}{l}}\right) + (l-0)\left(\frac{-\cos 0}{\frac{n^2\pi^2}{l^2}}\right)\right\}$

$a_n = \frac{2}{l}\left\{l\left(\frac{-l^2\cos n\pi}{n^2\pi^2}\right) + (l)\left(\frac{-l^2}{n^2\pi^2}\right)\right\}$

$a_n = \frac{-2l^3}{l\,n^2\pi^2}\left\{(-1)^n + 1\right\}$

Sub a_0 and a_n in (9), we get

. $lx-x^2=(l^2/3)\ /2+\sum_{n=1}^{\infty}\frac{-2l^2}{n^2\pi^2}\left\{(-1)^n+1\right\}\cos\frac{\pi nx}{l}$ -------(10)

Sub (10) in (8) , we get

$(l^2/3)/\ 2+\sum_{n=1}^{\infty}\frac{-2l^3}{l\,n^2\pi^2}\left\{(-1)^n+1\right\}\cos\frac{\pi nx}{l} = A_0+\sum_{n=1}^{\infty}A_n\cos\frac{\pi nx}{l}$

From the above by comparison, we get

$A_0=l^2/6$

$A_n = \frac{-2l^2}{n^2\pi^2}\left\{(-1)^n+1\right\}$

Sub A_0 and A_n in (7), we get

$u(x,t)=l^2/6+\sum_{n=1}^{\infty}\frac{-2l^2}{n^2\pi^2}\left\{(-1)^n+1\right\}\cos\frac{\pi nx}{l}\ e^{-\frac{\alpha^2\pi^2n^2t}{l^2}}$ ---------(7)

$u(x,t)=l^2/6-\frac{2l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{(-1)^n+1\right\}\cos\frac{\pi nx}{l}\ e^{-\frac{\alpha^2\pi^2n^2t}{l^2}}$

Which is the required solution

TWO ENDS ARE THERMALLY INSULATED

When the two ends x=0 and x=l of a rod of length l is thermally insulated then we have the following boundary conditions.

(i) $\frac{\partial u}{\partial x}_{at\ x=0} = 0$ **(ii)** $\frac{\partial u}{\partial x}_{at\ x=l} = 0$

Problem:-

Solve the equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ subject to the following conditions

(i) u is finite when t tend to infinity

(ii) $\frac{\partial u}{\partial x}_{at\ x=0} = 0$ for all t>0

(iii) u=0 when x=l, for all t>0

(iv) $u=u_0$ when t=0 for all values of x between 0 and l

Solution:-

The one dimensional heat flow equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$

Solving this equation by the method of separation of variable and applying conditions, we get the solution of the form

u(x,t)=(A cos px +B sin px) $e^{-\alpha^2 p^2 t}$-------(1)

Applying condition (ii) on (1),

Differentiate (1) partially w.r.t x, we get

$\frac{\partial u(x,t)}{\partial x}$=(-Ap sin px +Bp cos px) $e^{-\alpha^2 p^2 t}$

Sub x=0 on both sides , we get

$\frac{\partial u(0,t)}{\partial x}$=(-Ap sin 0 +Bp cos 0) $e^{-\alpha^2 p^2 t}$

0=Bp $e^{-\alpha^2 p^2 t}$

0=Bp [$e^{-\alpha^2 p^2 t}$ is non zero]

0=B [p cannot be zero]

Sub the value of B in (1), we get

u(x,t)=A cos px $e^{-\alpha^2 p^2 t}$ ----------(2)

Applying condition (iii) on (2), we get

Sub x=l in (2), we get

u(l,t)=A cos pl $e^{-\alpha^2 p^2 t}$

0=A cos pl $e^{-\alpha^2 p^2 t}$

0=A cos pl [$e^{-\alpha^2 p^2 t}$ is non zero]

0=cos pl [If A is zero ,it gives trivial solution]

$\cos\frac{(2n-1)\pi}{2}$ =cos pl, where n=1,2,3,......

$$pl = \frac{(2n-1)\pi}{2}$$

$$p = \frac{(2n-1)\pi}{2l}$$

Sub the value of p in (2), we get

u(x,t)=A cos $\frac{(2n-1)\pi x}{2l}$ $e^{-\frac{(2n-1)^2\alpha^2\pi t}{4l^2}}$ ---------(3)

The most general solution is given by

u(x,t)= $\sum_{n=1}^{\infty}$ A_n cos $\frac{(2n-1)\pi x}{2l}$ $e^{-\frac{(2n-1)^2\alpha^2\pi t}{4l^2}}$ ---------(4)

Applying condition (iv) on (4), we get

Sub t=0 in (4), we get

u(x,0)= $\sum_{n=1}^{\infty}$ A_n cos $\frac{(2n-1)\pi x}{2l}$ e^0

u_0= $\sum_{n=1}^{\infty}$ A_n cos $\frac{(2n-1)\pi x}{2l}$ --------(5)

Since the L.H.S of (5) is constant, to find the constant we use some fundamental calculus as follows

u_0= A_1 cos $\frac{\pi x}{2l}$ + A_1 cos + A_2 cos $\frac{3\pi x}{2l}$ + A_3 cos $\frac{5\pi x}{2l}$ +...............--------(6)

To find A_n

Multiply (6) by cos $\frac{(2n-1)\pi x}{2l}$ and integrate with respect to x from 0 to l, we get

$$u_0\int_0^l \cos\frac{(2n-1)\pi x}{2l}dx = A_1\int_0^l \cos\frac{\pi x}{2l}\cos\frac{(2n-1)\pi x}{2l}dx + A_2\int_0^l \cos\frac{3\pi x}{2l}\cos\frac{(2n-1)\pi x}{2l}dx$$

$$+ A_3\int_0^l \cos\frac{5\pi x}{2l}\cos\frac{(2n-1)\pi x}{2l}dx + \ldots\ldots\ldots A_n\int_0^l \cos\frac{(2n-1)\pi x}{2l}\cos\frac{(2n-1)\pi x}{2l}dx + \ldots\ldots$$

W.K.T $\int_0^l \cos mx \cos nx dx = 0$ if m is not equals to n

$$u_0\int_0^l \cos\frac{(2n-1)\pi x}{2l}dx = A_n\int_0^l \cos^2\frac{(2n-1)\pi x}{2l}dx$$

$$u_0\left(\frac{-\sin\frac{(2n-1)\pi x}{2l}}{\frac{(2n-1)\pi}{2l}}\right)_0^l = A_n\int_0^l [1+\cos\frac{(2n-1)\pi x}{l}]/2\,dx$$

$$u_0\left(\frac{-2l\sin\frac{(2n-1)\pi x}{2l}}{(2n-1)\pi}\right)_0^l = A_n/2\int_0^l 1+\cos\frac{(2n-1)\pi x}{l}dx$$

$$u_0\left(\frac{-2l}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2l}\right)_0^l = A_n/2\left\{(x)_0^l + \left(\frac{-\sin\frac{(2n-1)\pi x}{l}}{\frac{(2n-1)\pi}{l}}\right)_0^l\right\}$$

$$u_0\left(\frac{-2l}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2} + \frac{2l}{(2n-1)\pi}\sin 0\right) = A_n/2\left\{l + \left(\frac{-l\sin\frac{(2n-1)\pi x}{l}}{(2n-1)\pi}\right)_0^l\right\}$$

$$u_0\left(\frac{-2l}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2}\right) = A_n/2\left\{l + \left(\frac{-l}{(2n-1)\pi}\sin(2n-1)\pi + \frac{l}{(2n-1)\pi}\sin 0\right)\right\}$$

$u_0\left(\frac{-2l}{(2n-1)\pi}\sin\frac{(2n-1)\pi}{2}\right) = A_n/2\,\{l\}$

$$A_n = \frac{2}{l}\frac{-2lu_0}{(2n-1)\pi}(-1)^n$$

$$A_n = \frac{4u_0}{(2n-1)\pi}(-1)^{n+1}$$

Sub the value of A_n in (4) , we get

$$u(x,t)=\sum_{n=1}^{\infty} \frac{4u_0}{(2n-1)\pi}(-1)^{n+1}\cos\frac{(2n-1)\pi x}{2l}\, e^{-\frac{(2n-1)^2\alpha^2\pi t}{4l^2}}$$

Which is the required solution.